Study Guide for Exam 3 in MATH 2420

Section 4.3

Definitions, Theorems, etc from the section:

1. A real number r is **rational** is and only if it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational**. If r is a real number, then
$$r \in \mathbb{Q} \Leftrightarrow \exists\ a, b \in \mathbb{Z} \text{ such that } r = \frac{a}{b}, \text{ and } b \neq 0.$$
2. If neither of two real numbers is zero, then their product is also not zero.
3. Theorem: Every integer is a rational number.
4. Theorem: The sum of any two rational numbers is rational.
5. Be able to prove or use:
   (a) The sum, product, and difference of any two even integers are even.
   (b) The sum and difference of any two odd integers are even.
   (c) The product of any two odd integers is odd.
   (d) The product of any even integer and any odd integer is even.
   (e) The sum of any odd integer and any even integer is odd.
   (f) The difference of any odd integer minus any even integer is odd.
   (g) The difference of any even integer minus any odd integer is odd.
   (h) For any $r, s \in \mathbb{Q}$, $r + s$, $r - s$, $rs \in Q$ and $\frac{r}{s} \in \mathbb{Q}$ if $s \neq 0$.
6. A **corollary** is a statement whose truth can be immediately deduced from a theorem that has been proved.

Problems to be able to Solve:

1. Decide when to prove or disprove a statement involving rational numbers, integers, and even/odd numbers. Produce a proof or counterexample.
2. Use any of the above statements in a proof.
3. Represent any rational number as the ratio of two integers.
4. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 4.4

Definitions, Theorems, etc from the section:

1. If n and d are integers then, n **is divisible by** d if and only if n equals d times some integer and $d \neq 0$. Denoted $d|n$, symbolically

   $$d|n \Leftrightarrow \exists\ k \in \mathbb{Z} \text{ such that } n = dk \text{ and } d \neq 0.$$

   $d \nmid n$ denotes d does not divide n.

2. Theorem: For all integers a and b, if a and b are positive and a divides b, then $a \leq b$.
3. Theorem: The only divisors of 1 are 1 and -1.
4. For all integers n and d, $d \nmid n \Leftrightarrow \dfrac{n}{d} \notin \mathbb{Z}$.
5. Theorem: For all integers a, b, c, if $a \mid b$, $b \mid c$, then $a \mid c$.
6. Theorem: Any integer $n > 1$ is divisible by a prime number.
7. Theorem: Given any integer $n > 1$, there exist a positive integer k, distinct primes $p_1, \ldots, p_k$ and positive integers $e_1, \ldots, e_k$ such that

   $$n = p_1^{e_1} \cdots p_k^{e_k},$$

   and any other expression for n as a product of prime numbers is identical to this.

8. Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

   $$n = p_1^{e_1} \cdots p_k^{e_k},$$

   where k is a positive integer, $p_1, \ldots, p_k$ are distinct primes, and $e_1, \ldots, e_k$ are positive integers.

Problems to be able to Solve:

1. Decide when to prove or disprove a statement involving divisibility or prime. Produce a proof or counterexample.
2. Decide if an integer is divisible by another.
3. Prove properties of divisibility.
4. Give the prime factorization of an integer.
5. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 4.6

Definitions, Theorems, etc from the section:

1. Given any real number x, the **floor of** x, denoted $\lfloor x \rfloor$, is defined as

$$\lfloor x \rfloor = \text{ that unique integer } n \text{ such that } n \leq x < n+1.$$

2. Given any real number x, the **ceiling of** x, denoted $\lceil x \rceil$, is defined as

$$\lceil x \rceil = \text{ that unique integer } n \text{ such that } n-1 < x \leq n.$$

3. Exercise: If k is an integer, then $\lfloor k \rfloor = \lfloor k + 1/2 \rfloor = k$.
4. Theorem: For every real number x and every integer m, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.
5. Theorem: For every integer n , $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$ if n is even and $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$ if n is odd.
6. n div $d = \lfloor \frac{n}{d} \rfloor$.
7. n mod $d = n - d \cdot \lfloor \frac{n}{d} \rfloor$. Comment, these are the remainders when dividing n by d.
8. Theorem: If n is any integer and d is a positive integer, and if $q = \lfloor \frac{n}{d} \rfloor$ and $r = n - d \cdot \lfloor \frac{n}{d} \rfloor$ then

$$n = dq + r \text{ and } 0 \leq r < d.$$

Problems to be able to Solve:

1. Be able calculate floor and ceiling of any real number.
2. Be able to use the floor and ceiling functions in applications.
3. Prove or disprove statements involving the floor and ceiling.
4. Calculate div and mod.
5. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 5.2

Definitions, Theorems, etc from the section:

1. Principle of Mathematical Induction: Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

   (a) $P(a)$ is true.

   (b) For every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

   Then the statement, for every integer$n \geq a$, $P(n)$ is true.

2. Method for proving using mathematical induction:

   (a) Base Case. show $P(a)$ is true for some a.

   (b) Induction: Assume that for every integer $k \geq a$ that $P(k)$ is true, show that $P(k+1)$ is true.

3. Theorem: For ever integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

4. If a sum with a variable number of terms is shown to equal an expression that does not contain either an ellipsis or a summation symbol, we say that the sum is written **in closed form**. Example is the above theorem.

5. For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Problems to be able to Solve:

1. Be able to use the theorems proven in the section to answer questions.

2. Be able to set up the steps for induction, which include stating or verifying the base case, stating $P(k)$ and $P(k+1)$ for a property $P(n)$.

3. Prove statements that use the method of induction.

4. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 6.1

Definitions, Theorems, etc from the section:

1. If S is a set, the notation $x \in S$ means that x is an element of S. The notation $x \notin S$ means that x is not an element of S.
2. A set S may be specified in **set-roster notation** meaning that all the elements in S are written between braces $\{...\}$.
3. **Axiom of Extension:** a set is completely determined by what its elements are, not the order nor whether or not they are repeated more than once.
4. $\mathbb{R}$ represents the set of all real numbers. Note that we say "nonneg" in the superscript to mean we include zero, and + if we want the numbers strictly greater than 0.
5. $\mathbb{Z}$ represents the set of all integers. Note that we say "nonneg" in the superscript to mean we include zero, and + if we want the numbers strictly greater than 0.
6. $\mathbb{Q}$ represents the set of rational numbers. Note that we say "nonneg" in the superscript to mean we include zero, and + if we want the numbers strictly greater than 0.
7. Let S denoted a set and $P(x)$ be some property that the elements of S may or may not apply. The set defined as **the set of all elements $x \in S$ such that $P(x)$ is true** is denoted by
$$\{x \in S | P(x)\}.$$
8. Given two sets A and B, we say that A **is a subset of** B, $A \subseteq B$ if and only if every element of A is also in B, i.e. $A \subseteq B$ means if $x \in A$ then $x \in B$.
9. Given two sets A and B, we say that A **is a proper subset of** B, $A \subset B$ if and only if every element of A is also in B and there exists at least one element of B not in A, i.e. $A \subset B$ means if $x \in A$ then $x \in B$ and there exists an $y \in B$ such that $y \notin A$.
10. Given a positive integer n and let $x_1, \ldots, x_n$ be elements. The **ordered n-tuple** $(x_1, \ldots, x_n)$ consists of $x_1, \ldots, x_n$ together with an ordering, x_1 first, x_2 second, and so on. Remember that $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, means that $x_1 = y_1$, $x_2 = y_2$, and so on.
11. $A \subseteq B$ if and only if $\forall x$, if $x \in A$ then $X \in B$.
12. $A \nsubseteq B$ if and only if $\forall x$, if $x \in A$ then $X \in B$.
13. A is a **proper subset of** B if and only if $A \subseteq B$ and there exists at least on element in B that is not in A.
14. Element Arguments: Lets X and Y be given. To prove $X \subseteq Y$

(a) Suppose that x is a particular but arbitrary chosen element of X.

(b) Show that x is an element of Y.

15. Given sets A and B, A **equals** B, $A = B$, if and only if every element of A is in B and every element of B is in A. Meaning $A \subseteq B$ AND $B \subseteq A$.

16. A **universal set** is a set that contains every element in question, also called a **universe of discourse**.

17. The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in A OR B.

$$A \cup B = \{x \in U | x \in A \text{ or } x \in B\}$$

18. The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are in A AND B.

$$A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$$

19. The **difference** of B minus A, denoted $B - A$, is the set of all elements that are in B and not in A.

$$B - A = \{x \in U | x \notin A \text{ and } x \in B\}$$

20. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

$$A^c = \{x \in U | x \in U \text{ and } x \notin A\}$$

21. Interval Notation:

    (a) $(a, b) = \{x \in \mathbb{R} | a < x < b\}$

    (b) $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

    (c) $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$

    (d) $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

    (e) $(a, \infty) = \{x \in \mathbb{R} | x > a\}$

    (f) $[a, b) = \{x \in \mathbb{R} | x \geq a\}$

    (g) $(-\infty, b) = \{x \in \mathbb{R} | x < b\}$

    (h) $(\infty, b] = \{x \in \mathbb{R} | x \leq b\}$

22. (a) $\bigcup_{i=0}^{n} A_i = \{x \in U | x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$

    (b) $\bigcup_{i=0}^{\infty} A_i = \{x \in U | x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots\}$

    (c) $\bigcap_{i=0}^{n} A_i = \{x \in U | x \in A_i \text{ for every } i = 0, 1, 2, \ldots, n\}$

    (d) $\bigcap_{i=0}^{\infty} A_i = \{x \in U | x \in A_i \text{ for every } i = 0, 1, 2, \ldots, n\}$

23. The **empty set** is the set with no elements, denoted $\emptyset$.

24. Two sets are called **disjoint** if and only if they have no elements in common, i.e. their intersection is the empty set.

$$A \text{ and } B \text{ are disjoint } \Leftrightarrow A \cap B = \emptyset$$

25. Sets $A_1, A_2, \ldots$ are **mutually disjoint** if and only if there are no two sets A_i and A_j with distinct subscripts have any elements in common.

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j$$

26. A finite or infinite collection of nonempty sets $\{A_1, A_2, \ldots\}$ is a **partition** of a set A if and only if A is the union of all of the A_i and the sets A_1, $A_2, \ldots$ are mutually disjoint.

27. Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A. Note, this includes the empty set and A itself.

Problems to be able to Solve:

1. Show if two sets are subsets or equal by checking every element or using a particular element.
2. Use Venn Diagrams to find areas denoted by a specific set.
3. Perform any of the set operations on given sets.
4. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 6.2

Definitions, Theorems, etc from the section:

1. Theorem: For all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
2. Theorem: For all sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
3. Theorem: For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
4. Procedural Versions of Set Definitions
   (a) $x \in X \cup Y$ if and only if $x \in X$ or $x \in Y$.
   (b) $x \in X \cap Y$ if and only if $x \in X$ and $x \in Y$.
   (c) $x \in X - Y$ if and only if $x \in X$ and $x \notin Y$.

(d) $x \in X^c$ if and only if $x \notin X$.

(e) $(x, y) \in X \times Y$ if and only if $x \in X$ and $y \in Y$.

5. An **identity** is an equation that is universally true for all elements in some set.

6. For all sets A, B, and C. Let U be the universal set.

   (a) Commutative Laws: $A \cup B = B \cup A$, and $A \cap B = B \cap A$.

   (b) Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

   (c) Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (d) Identity Laws: $A \cup \emptyset A$ and $A \cap U = A$.

   (e) Complement Laws: $A \cup A^c = U$ and $A \cap A^c = \emptyset$.

   (f) Double Complement Law: $(A^c)^c = A$.

   (g) Idempotent Laws: $A \cup A = A$ and $A \cap A = A$.

   (h) Universal Bound Laws: $A \cup U = U$ and $A \cap \emptyset = \emptyset$.

   (i) DeMorgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

   (j) Absorption Laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

   (k) Complements of U and $\emptyset$: $U^c = \emptyset$ and $\emptyset^c = U$.

   (l) Set Difference Law: $A - B = A \cap B^c$.

7. Theorem: The empty set is unique and is a subset of any set.

8. Theorem: $A \cup (\bigcap_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} (A \cup B_i)$.

Problems to be able to Solve:

1. Be able to prove or disprove statements using the procedural definitions of the set operations.

2. Be able to prove or disprove statements using the set identities.

3. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 7.1

Definitions, Theorems, etc from the section:

1. A **function** f from a set X to a set Y, denoted $f : X \to Y$, is a relation from X, the **domain** of f to Y, the **co-domain** of f that satisfies two properties.

(a) every element in X is related to some element in Y.

(b) no element in X is related to more than one element in Y.

Meaning that for any element $x \in X$ there is a unique element $y \in Y$ such that $f(x) = y$.

2. **range of f= image of X under** $f = \{y \in Y | y = f(x) \text{ for some } x \in X\}$

3. **inverse image of** f= $f^{-1}(y) = \{x \in X | f(x) = y\}$.

4. We can use arrow diagrams to denote functions from X to Y. All we need to check is that every element in X has an arrow that points to an element in Y and no element in X has two arrows that point to two different elements in Y.

5. Theorem: If $F : X \to Y$ and $G : X \to Y$ are functions, then $F = G$ if and only if $F(x) = G(x)$ for every $x \in X$.

6. The **identity function** on a set X, denoted I_X from X to X, is defined as $I_X(x) = x$ for each $x \in X$.

7. A function defined on the power set of A, $F : \mathcal{A} \to \mathbb{Z}^{nonneg}$ defined by $F(X) =$ the number of elements in X for $X \subseteq A$.

8. Functions defined on the a Cartesian product. We can define $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $M(a, b) = ab$ and $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times R$ by $R(a, b) = (-a, b)$. Make sure you remember what each function has as an output.

9. Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of** x, written $\log_b x$, is the exponent to which b must be raised to obtain x.

$$\log_b x = y \Leftrightarrow \; b^y = x.$$

This is a function from $\mathbb{R}^+$ to $\mathbb{R}$.

10. Useful log properties, note I am dropping the base in these but the bases need to be the same for these to work.

(a) $\log(ab) = \log a + \log b$

(b) $\log(a/b) = \log a - logb$

(c) $\log(a^b) = b \log a$

11. An **n-place Boolena function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$.

12. A function is **not well-defined** if it fails to satisfy at least one of the requirements of being a function.

13. If $f : X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y | y = f(x) \text{ for some } x \in A\}$$

$$f^{-1}(C) = \{x \in X | f(x) \in C\},$$

where $f(A)$ is called the **image of** A and $f^{-1}(C)$ is called the **inverse image of** C.

Problems to be able to Solve:

1. Determine if an arrow diagram is a function.
2. Use an arrow diagram to find the domain, co-domain, range, image, pre-image, and etc for a function.
3. Represent a function as a set of ordered pairs.
4. Determine if two given functions are equal.
5. Use any of the above functions to find the outputs for a given domain element.
6. Be able to evaluate a logarithm.
7. Find mod and div and be able to use them in functions.
8. Be able to solve any question similar to lecture, homework, or an in-class assignment.