Study Guide for Exam 4 in MATH 2420

Section 7.1

Definitions, Theorems, etc from the section:

1. A **function** f from a set X to a set Y, denoted $f : X \to Y$, is a relation from X, the **domain** of f to Y, the **co-domain** of f that satisfies two properties.

   (a) every element in X is related to some element in Y.

   (b) no element in X is related to more than one element in Y.

   Meaning that for any element $x \in X$ there is a unique element $y \in Y$ such that $f(x) = y$.

2. **range of** f**= image of** X **under** $f = \{y \in Y | y = f(x) \text{ for some } x \in X\}$

3. **inverse image of** f= $f^{-1}(y) = \{x \in X | f(x) = y\}$.

4. We can use arrow diagrams to denote functions from X to Y. All we need to check is that every element in X has an arrow that points to an element in Y and no element in X has two arrows that point to two different elements in Y.

5. Theorem: If $F : X \to Y$ and $G : X \to Y$ are functions, then $F = G$ if and only if $F(x) = G(x)$ for every $x \in X$.

6. The **identity function** on a set X, denoted I_X from X to X, is defined as $I_X(x) = x$ for each $x \in X$.

7. A function defined on the power set of A, $F : \mathcal{A} \to \mathbb{Z}^{nonneg}$ defined by $F(X) =$ the number of elements in X for $X \subseteq A$.

8. Functions defined on the a Cartesian product. We can define $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $M(a, b) = ab$ and $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times R$ by $R(a, b) = (-a, b)$. Make sure you remember what each function has as an output.

9. Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base** b **of** x, written $\log_b x$, is the exponent to which b must be raised to obtain x.

   $$\log_b x = y \Leftrightarrow \ b^y = x.$$

   This is a function from $\mathbb{R}^+$ to $\mathbb{R}$.

10. Useful log properties, note I am dropping the base in these but the bases need to be the same for these to work.

(a) $\log(ab) = \log a + \log b$

(b) $\log(a/b) = \log a - logb$

(c) $\log(a^b) = b \log a$

11. An **n-place Boolena function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$.

12. A function is **not well-defined** if it fails to satisfy at least one of the requirements of being a function.

13. If $f : X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y | y = f(x) \text{ for some } x \in A\}$$

$$f^{-1}(C) = \{x \in X | f(x) \in C\},$$

where $f(A)$ is called the **image of** A and $f^{-1}(C)$ is called the **inverse image of** C.

Problems to be able to Solve:

1. Determine if an arrow diagram is a function.
2. Use an arrow diagram to find the domain, co-domain, range, image, pre-image, and etc for a function.
3. Represent a function as a set of ordered pairs.
4. Determine if two given functions are equal.
5. Use any of the above functions to find the outputs for a given domain element.
6. Be able to evaluate a logarithm.
7. Find mod and div and be able to use them in functions.
8. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 7.2

Definitions, Theorems, etc from the section:

1. Let F be a function from a set X to a set Y, F is **one-to-one**, **injective** if and only if for all elements $x_1, x_2 \in X$, if $F(x_1) = F(x_2)$, the $x_1 = x_2$. Equivalently, $x_1 \neq x_2$ implies that $F(x_1) \neq F(x_2)$.

Comment, this has to hold for all pairs of x_1 and x_2, if one pair fails, then the function is not injective.

2. A function $F : X \to Y$ is not injective if there exists a x_1 and x_2 where $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

3. Let F be a function from a set X to a set Y, F is **onto**, **surjective** if and only if given any element in $y \in Y$, it is possible to find an element $x \in X$ such that $F(x) = y$.

   This has to hold for every element in Y. Another way to think f this is that range(F)=co-domain of F.

4. Useful log properties, note the bases need to be the same for these to work.

   (a) $\log_x(ab) = \log_x a + \log_x b$
   (b) $\log_x(a/b) = \log_x a - \log_x b$
   (c) $\log_x(a^b) = b \log_x a$
   (d) $\log_x c = \dfrac{\log_b c}{\log_b x}$

5. Useful exp properties, note the bases need to be the same for these to work except for the last one.

   (a) $b^u b^v = b^{u+v}$
   (b) $(b^u)^v = b^{uv}$
   (c) $\dfrac{b^u}{b^v} = b^{u-v}$
   (d) $(bc)^u = b^u c^u$.

6. A **one-to-one correspondence**, **bijection**, from a set X to a set Y is a function that is one-to-one and onto.

7. Theorem: Suppose that $F : X \to Y$ is a one-to-one correspondence. Then there is a function $F^{-1} : Y \to X$ that is defined by, given any $y \in Y$, $F^{-1}(y) =$ unique $x \in X$ such that $F(x) = y$.

8. The F^{-1} defined previously is called the **inverse of** F.

   Comment: To find the inverse's arrow diagram just flip the errors. If it is a function, flip the x's and y's and resolve for y.

9. Theorem: If X and Y are sets and $F : X \to Y$ is one-to-one and onto, then $F^{-1} : Y \to X$ is one-to-one and onto.

Problems to be able to Solve:

1. Recognize definitions and answer true-false statements.
2. Show and or determine if a function is one-to-one, onto, or a bijection.
3. Find the inverse of a given functions and its arrow diagram.

Section 8.1

Definitions, Theorems, etc from the section:

1. Let A and B be sets. A **relation R** from A to B is a subset of $R \subseteq A \times B$. Given an ordered pair $(x, y) \in A \times B$, x is related to y be E, written xRy is and only if $(x, y) \in R$. The set A is the domain of R and the set B is the co-domain. xRy if and only if $(x, y) \in R$. Moreover x is not related to y by R if and only if $(x, y) \notin R$.
2. Let R be realtion from A to B, the inverse relation R^{-1} from B to A as
$$R^{-1} = \{(y, x) \in B \times A | (x, y) \in R\}$$
3. Let R be relation, the inverse of the relation, R^{-1} is found be interchanging the elements in the ordered pair, i.e. $(x, y) \in R$ if and only if $(y, x) \in R^{-1}$. Also done by reversing the arrows in the arrow diagram.
4. A **relation on a set** A is a relation from A to A.
5. A relation on a set A produces a directed graph when the arrow diagrams is created.
6. Given sets $A_1, A_2, A_3, \ldots, A_n$ an **n-ary relation** R on $A_1 \times \cdots \times A_n$ is a subset of $A_1 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary**, and **quaternary** relations, respectively.

Problems to be able to Solve:

1. Determine if two elements are related.
2. Draw the arrow diagrams for a relation.
3. Draw the directed graph for a relation on a set.
4. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 8.2

Definitions, Theorems, etc from the section:

1. Let R be a relation of a set A
   (a) R is **reflexive** if and only if for every $x \in A$, xRx.
   (b) R is **symmetric** if and only if for every $x, y \in A$, if xRy, then yRx.
   (c) R is **transitive** if and only if for every $x, y, z \in A$, if xRy and yRz, then xRz.

   Note, R is not reflexive if there exists and $x \in A$ such that x is not related to x by R.

   Note, R is not symmetric if there exists an $x, y \in A$ such that xRy AND y is NOT related to x, i.e. yRx is FALSE.

   Note, R is not transitive if there exists an $x, y, z \in A$ such that xRy, yRz, and x is NOT related to z by R, i.e. xRz is FALSE.
2. Equality relation, xRy if any only if $x = y$, this is reflexive, symmetric, and transitive.
3. Less Than relation, xRy if and only if $x < y$ is transitive, NOT symmetric, and NOT reflexive.
4. Congruence Modulo d, xRy if and only if $d|(x-y)$, this is reflexive, symmetric, and transitive.
5. Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the properties
   (a) R^t is transitive
   (b) $R \subseteq R^t$
   (c) If S is any other transitive relation that contains R, then $R^t \subseteq S$. Meaning R^t is the smallest.

Problems to be able to Solve:

1. Show if a relation is reflexive, transitive, and or symmetric.
2. Use any of the relations mentioned in class or homework.
3. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 8.3

Definitions, Theorems, etc from the section:

1. A finite or infinite collection of nonempty sets $\{A_1, A_2, \ldots\}$ is a **partition** of a set A if and only if A is the union of all of the A_i and the sets A_1, $A_2, \ldots$ are mutually disjoint.

2. Given a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows, for every $x, y \in A$, xRy if and only if there is a subset A_i of the partition such that both x and y are in A_i.

3. Theorem: Let A be a set with a partition and let R be a relation induced by the partition. Then R is reflexive, symmetric, and transitive.

4. Let A be a set and R a relation on A. R is an **equivalence relation** if and only if R is reflexive, symmetric, and transitive.

5. Suppose A is a set and R an equivalence relation on A. For each element $a \in A$, the **equivalence class of** a, denoted $[a]$m is the set of all elements in A such that xRa,

$$[a] = \{x \in A | xRa\}.$$

6. Relation on Subsets, given a set X and $A, B \in \mathcal{P}(X)$, then ARB if and only if they have the same least element.

7. Lemma: Suppose A is a set and R is an equivalence relation on A, for $a, b \in A$, if aRb then $[a] = [b]$.

8. Lemma: If A is a set with R an equivalence relation on A. Then for any $a, b \in A$, either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

9. Theorem: If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A.

10. Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element $a \in A$ such that $[a] = S$, meaning $a \in S$ as a set.

11. Let m and n be integers and d a positive integer. We say that **m is congruent to n modulo d** and write $m \equiv n \mod d$ if and only if $d | (m - n)$.

    Comment, the modulo relation on the integers is an equivalence relation, moreover the equivalence classes are the remainders when dividing by d, $[0], [1], \ldots, [d-1]$.

12. Rational relation, $A = \mathbb{Z} \times \mathbb{Z}$, for all pairs $(a, b), (c, d) \in A$, $(a, b)R(c, d)$ if and only if $ad = bc$.

Problems to be able to Solve:

1. Determine if a relation is an equivalence relation.
2. Find the relation if given a partition.
3. Find the partition if given a relation.
4. Find the equivalence classes.
5. Use any of the relations mentioned.
6. Calculations use congruence modulo d.
7. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 10.1

Definitions, Theorems, etc from the section:

1. A **graph** G consists of two finite sets, a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges where each edge is associated with a set consisting of either one or two vertices called **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function.**

   An edge with one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edges is said to connect its two endpoints; two vertices that are connected are called **adjacent**, and a vertex that is the endpoint of a loop is **adjacent to itself.**

   An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex that is incident to no edge is called **isolated**.

2. A **walk from** v to w is a finite alternating sequence of adjacent edges and vertices.
3. A **trivial walk** from v to v consists of the single vertex v.
4. A **trail from** v to w is a walk that DOES NOT contain a repeated edge.
5. A **path from** v to w is a TRAIL that ALSO DOES NOT contain a repeated vertex. These have no repeated vertices AND edges.
6. A **closed walk** is a walk that starts and ends at the same vertex.
7. A **circuit** is a CLOSED WALK that has AT LEAST one edge and has NO repeated edges.

8. A **simple circuit** is a CIRCUIT that DOES NOT have any repeated vertices EXCEPT the first and last.

9. A graph H is a **subgraph** of a graph G if and only if every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

10. Let G be a graph. Two vertices v and w are **connected (vertices)** if and only if there is a walk from v to w. The graph G is **connected (graph)** if and only if given ANY two vertices v and w in G there is a walk from v to w.

11. Lemma: Let G be graph.

    (a) If G is connected, then any two distinct vertices of G can be connected by a path.

    (b) If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there exists a trail from v to w in G.

    (c) If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

12. A graph H is a **connected component** of G if and only if

    (a) H is a subgraph of G,

    (b) H is connected,

    (c) no connected subgraph of G has H as a subgraph and contains vertices or edges not in H.

13. Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge in G.

14. Theorem: If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has a Euler circuit.

15. Theorem: IF G has a vertex of odd degree then the graph does not have an Euler circuit.

16. Comment: Degree is the number of edges connected to the vertex, i.e. incident with. Note a loop counts twice.

17. Algorithm for Euler Circuits:

    (a) Pick any vertex of G.

    (b) Pick any sequence of adjacent edges and vertices, starting and ending at the same vertex and never repeating an edge. Call the resulting circuit C.

    (c) Check if C contains every edge, if so you are done.

i. If not, remove all edges of C from G and any vertices that become isolated, call the resulting subgraph G'.
ii. Pick any vertex in common to both C and G', saw w.
iii. Pick any sequence of adjacent vertices and edges in G' starting and stopping at w and never repeating an edge. Call the resulting circuit C'.
iv. Patch C and c' together to create a circuit C''. Start at v, follow C to w, then follow C' back to x, and travel the rest of C back to v.
v. Let $C = C''$ and start at (i.) again.

Problems to be able to Solve:

1. Identify walks, paths, trails, etc on a graph.
2. Determine if a graph is a subgraph.
3. Find all possible subgraphs of a given graph.
4. Identify if a graph is connected.
5. Identify connected components.
6. Determine if a graph has an Euler circuit, and find it.
7. Be able to solve any question similar to lecture, homework, or an in-class assignment.

Section 10.2

Definitions, Theorems, etc from the section:

1. An $m \times n$ matrix A over a set S is a rectangular array of elements of S arranged into m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

We write $A = (a_{ij})$ where a_{ij} refers to the entry in the ith row and jth column.

2. An $m \times n$ matrix A has **size** $m \times n$.

3. Matrices A and V are equal if and only if they have the same size and the corresponding entries are the same.

4. A matrix is square if the number of columns equals the number of rows.

5. The main diagonal of A are the entries $a_{11}, \ldots, a_{nn}$ for a square $n \times n$ matrix.

6. A **directed graph** consists of two finite sets, a nonempty set of vertices $V(G)$ and a set $D(G)$ of directed edges where each is associated with a pair of vertices called it endpoints. If edge e is associated with the pair (v, w) of vertices, then e is the directed edge from v to w.

7. Let G be a directed graph with ordered vertices $v_1, \ldots, v_n$. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \# \text{ of arrows from } v_i \text{ to } v_j.$$

8. Let G be an undirected graph with ordered vertices $v_1, \ldots, v_n$. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \# \text{ of edges connecting } v_i \text{ to } v_j.$$

9. An $n \times n$ matrix A is **symmetric** if and only if for every $i, j = 1, \ldots, n$ we have $a_{ij} = a_{ji}$.

10. Let G be a graph with connected components $G_1, \ldots, G_k$. If there are n_i vertices in each connected component G_i and these vertices are numbered consecutively, then the adjacency matrix of G has the form

$$\begin{bmatrix} A_1 & \mathrm{O} & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{O} \\ 0 & A_2 & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{O} \\ 0 & \mathrm{O} & A_3 & \cdots & \mathrm{O} & \mathrm{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \mathrm{O} & \mathrm{O} & 0 & \cdots & \mathrm{O} & A_k \end{bmatrix}$$

where each A_i is $n_i \times n_i$ adjacency matrix of G_i for every i and the O's represent matrices with all 0 entries.

11. Suppose that all entries in matrices A and B are real numbers. If the number of elements, n, in the ith row of A equals the number of elements in the jth column of B, then the **scalar product** or **dot product** of the ith row of A with the jth column of B is the real number obtained by

$$a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

.

12. To multiply matrices A and B to create AB, A must have size $m \times n$ and B must have size $n \times p$, to create the matrix AB of size $m \times p$.

13. Matrix multiplication is not commutative, i.e. $AB \neq BA$ in general.
14. Matrix multiplication is associative.
15. The identity matrix is the $n \times n$ matrix whose main diagonal is all 1's and everything else is 0.

Problems to be able to Solve:

1. Find the adjacency matrix of a graph.
2. Determine if a graph has connected components from the adjacency matrix.
3. Multiply two matrices together.