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MATH 1001
Quantitative Reasoning

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The purpose of these notes is to provide a concise overview of the material covered in MATH 1001 at Georgia State University. Note that the chapters are presented in order that they are presented in the course due to exam scheduling.

Chapter 3

This chapter discusses topics related to logic which includes statements, quantifiers, truth tables, and more.

3.1

Definition 1. *A **statement** is a sentence that is true or false, but not both simultaneously.*

We deal with statements all the time, for example Ätlanta is the capital of Georgiaïs a statement that is true. Also, "the Broncos drafted Elway in 1983ïs also a statement, but in this case it is a false statement.

However, things that are not statements are of the form

1. Commands, "think before you act."
2. Questions, "does the restaurant have a bathroom?"
3. Opinions, "BBQ is the best food."

To save us from having to constantly rewrite a statement it is common to name our statements, typically with a p, q, r, s. Based on our first example we could say

p: Atlanta is the capital of Georgia.

Similar to sets, there are operations we can do to statements, the first one is negation. Where the **negation** of a statement is the process of changing a true statement false or a false statement true, this is denoted $\sim p$.

Example 2. *Given the statements p: Atlanta is the capital of Georgia, and q: the Broncos drafted Elway in 1983.*

$\sim p$*: Atlanta is not the capital of Georgia.*
$\sim q$*: The Broncos did not draft Elway in 1983.*

The thing to realize is that we want to change the statements p and q in such a way that it does not change the original statement. This is typically done by using the words not or is.

In English we encounter words like all, some, no, and none, these words are called **quantifiers**. Any statement with a quantifier is called a **quantified statement**. Some examples of quantified statements are:

1. All dogs are animals.
2. Some people are bigots.
3. No cats are dogs.
4. Some rectangles are squares.

There are other ways we can write these statements that would make better grammatical sense.

Original	Alternate	Example
All A are B.	There are no A that are not B.	There are no dogs that are not animals.
Some A are B.	There exists at least one A that is B.	There exists at least one person who is a bigot.
No A are B.	All A are not B.	All cats are not dogs.
Some A are not B.	Not all A are B.	Not all rectangles are squares.

When it comes to negation, negating quantified statements is tricky due to the quantifiers, however there is an easy way to remember the negation for each quantifier.

	Negation	
All A are B	$\Leftrightarrow$	Some A are not B
Some A are B	$\Leftrightarrow$	No A are B.

Let us consider some examples using the statements

p: All cats are brown.	$\sim p$: Some cats are not brown.
q: Some cats are not polydactyl.	$\sim q$: All cats are polydactyl.
r: Some cats are fluffy.	$\sim r$: No cats are fluffy.

Lastly, negating a negation returns original statement, symbolically this is $\sim\sim p = p$. So the negation of the statement ßome cats are not brown,ïs äll cats are brown."

3.2

In this section we will start combining "simple" statements to make more complex statements. Formally a **simple statement** conveys one idea without connecting words. On the other hand, a **compound statement** is a statement that combines two or more simple statements using connectives. These connectives are: and, or, if — ,then —, and if and only if.

For the remainder of this sections we will assume that p and q are simple statements. Here we will let p : It is Sunday, and q : The Broncos are playing. We will look at a few examples using the and connective and the previous operation, negation.

Definition 3. *The compound statement p and q, symbolized $p \wedge q$, is called a* ***conjunction****. The symbol $\wedge$ represents the word and.*

$p \wedge q$: It is Sunday and the Broncos are playing.
$p \wedge \sim q$: It is Sunday and the Broncos are not playing.
$\sim p \wedge q$: It is not Sunday and the Broncos are playing.
$\sim p \wedge \sim q$: It is not Sunday and the Broncos are not playing.

There are other words that we can use instead of and, the most common ones are but, yet, and nevertheless.

The next operation involves the connective or. More precisely, two simply statements connected by or. An example of this I have a cat or a dog. However an or statement can take on the one of two versions

1. Inclusive or: I have a cat or a dog or both.
2. Exclusive or: I have a cat or dog, but not both.

An inclusive or allows for both part to happen simultaneously, while an exclusive or does not.

Definition 4. *A compound statement with an inclusive or is called a* ***disjunction****, denoted $p \vee q$. Here $\vee$ represents the word or.*

Let us look at some examples using r : the bill receives majority approval, and s : the bill becomes law.

$p \vee q$: The bill receives majority approval or the bill becomes law.
$p \vee \sim q$: The bill receives majority approval or the bill does not become law.
$\sim p \vee q$: The bill does not receive majority approval or the bill becomes law.
$\sim p \vee \sim q$: The bill does not receive majority approval or the bill does not become law.

The next type of compound statement uses the if p, then q connective.

Definition 5. *The compound statement, if p, then q, denoted $p \to q$ is called a* **conditional** *statement. Here p is called the antecedent and q is the consequent.*

Consider the following simple statements, p : a person is human and q : a person is from earth.

$p \to q$: If a person is human, then that person is from earth.
$p \to \sim q$: If a person is human, then that person is not from earth.
$\sim p \to q$: If a person is not human, then that person is from earth.
$\sim p \to \sim q$: If a person is not human, then that person is not from earth.

Similar to the other connectives, there are other options for expressing conditional statement which include q if p, p is sufficient for q, or p only if q.

The last type of statement is defined as

Definition 6. ***Biconditional*** *statements are conditional statements that are true if the antecedent and consequent are reversed, denoted $p \leftrightarrow q$. A biconditional is a statement where $p \to q$ and $q \to p$ are both true.*
A biconditional statement is a compound statement where the connective is p if and only if q, sometimes this is represented by the shorthand p iff q.

If we consider the statement, if you have a cat, then you have a pet. We see that the statement is true for the current choice of antecedent (you have a cat) and consequent (you have a pet). However if you reverse them to create the condition statement, if you have a pet, then you have a cat is not true, i.e. you could have a dog. This is NOT a biconditional statement.

However, consider the conditional statement, if you have a cat, then you have a feline pet. This is also a biconditional statement since if you have a feline pet, then you have a cat is also true. Therefore we could say, you have a cat if and only if you have a feline pet.

When dealing with symbolic forms of our statements, we are able to place parentheses, and depending on the placement the meaning can change. For example, consider p : she is happy and s : she is wealthy.

$\sim (p \wedge q)$: It is not true that she is happy and wealthy.
$\sim p \wedge q$: She is not happy and she is happy.
$\sim (p \vee q)$: She is neither happy nor happy.
$\sim p \vee q$: She is not happy or she is wealthy.

When going from symbols to English we have a couple of rules of thumb.

The simple statements in parentheses come before the comma.

For example $(q \wedge \sim p) \to r$ can we written as “If q and not p, then not r .” Similarly, $q \vee (\sim p \to \sim r)$ can be written as “q, and if not p, then not r.”

In the case that we have a symbolic statement without parentheses, we have something called dominance order. Specifically, if a symbolic statement appears without parentheses, statements before and after the most dominant connective are grouped together. The **dominance order** is

Weakest						Strongest
Negation	-	Conjunction Disjunction	-	Conditional	-	Biconditional

Symbols	Dominant Symbol	Clarified Statement
$p \to q \wedge \sim r$	$\to$	$p \to (q \wedge \sim r)$
$p \wedge q \to \sim r$	$\to$	$(p \wedge q) \to \sim r$
$p \leftrightarrow q \to r$	$\leftrightarrow$	$(p \leftrightarrow q) \to r$
$p \to q \leftrightarrow r$	$\leftrightarrow$	$p \to (q \leftrightarrow r)$
$p \wedge \sim q \to r \vee s$	$\to$	$(p \wedge \sim q) \to (r \vee s)$
$p \wedge q \vee r$	Ambiguous	

For the final example, let p : I fail the course, q : I study hard, and r : I pass the final. Express in a symbolic form.

Ï do not pass the course if and only if I study hard and pass the final."

Looking at the statement we see that there is a negation of p and q, r appear as normal. Therefore we could write the statement as $\sim p \leftrightarrow q \wedge r$, however if we look at the dominant connective it is the biconditional. Therefore we can represent the statement as

$$\sim p \leftrightarrow (q \wedge r).$$

By a similar argument, consider the statement Ï do not pass the course if and only if I study hard, and I pass the final."

The thing to note is that there is a comma, so we can use a combination of the comma hint from earlier as well as the dominance of the biconditional. Therefore we can write the symbolic statement as

$$(\sim p \leftrightarrow q) \wedge r.$$

3.3

In this section we will start assigning truth values to our statements. More precisely, if we know if simple statements are true or false, then we can determine whether a statement involving negation, conjunctions, and disjunctions are true or false.

We are able to do this through truth tables, which is a method of see how the different connectives (symbols) affect every possible combination of true and false. We will start with negation, remember that negation changes a true statement false and a false statement true. Therefore the truth table for negation is

p	$\sim p$
T	F
F	T

For the conjunction, let us consider the statement: I visited London and Paris. This statement can be broken up into two simple statements p : I visited London and q : I visited Paris. Whether or not the conjunction is true can be determined by considering the possible combinations of the simple statements p and q.

Case	p	q	$p \wedge q$
Both London and Paris are visited.	T	T	T
Visited only London, but not Paris.	T	F	F
Visited only Paris, but not London.	F	T	F
Did not visit either cities.	F	F	F

For a conjuction, the only time the statement was true was when both cities were visited. This make sense since "and" statements requires both things to happen for it to be true. Therefore the conjunction truth table is given by:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For a disjunction, consider the statement: I visited London or Paris. If we remember that this is an inclusive or, then this means I visited London or Paris or both. So the question now is, when is this statement false? There is an easy answer, the statement is only false if I did not visit London and I did not visit Paris. Therefore we can find the truth table for the disjunction as:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 7. *Consider the statements*

$$p : 11 > 4 \qquad q : 3 < 5,$$

then determine if the following are true or false. Note that as stated, both p and q are true as stated.

$p \wedge q$ *: Both are p and q are true, and by the conjunction table we see that $p \wedge q$ is true.*

$\sim p \wedge q$ *: Notice that $\sim p$ is now false and q is true, then by the conjunction table we have that $\sim p \wedge q$ is false.*

$p \vee \sim q$ *: Notice that $\sim q$ is now false and p is true, then by the disjunction table we have that $p \vee \sim q$ is true.*

$\sim p \vee \sim q$ *: Notice that $\sim q$ and $\sim p$ are both now false, then by the disjunction table we have that $\sim p \vee \sim q$ is false.*

In the case that we have a statement that all possible combinations of true and false yield that the statement is true, then we call that statement a **tautology**. The easiest tautology to write is $p \vee \sim p$.
Let us construct the truth table for $(\sim p \vee q) \wedge \sim q$.

p	q	$\sim p$	$\sim p \vee q$	$\sim q$	$(\sim p \vee q) \wedge \sim q$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	F	F
F	F	T	T	T	T

Let us produce a truth table that involves three simple statements which is the most we will deal with during this course.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

3.4

In this section we will introduce the conditional and biconditional truth tables. First we will look at the conditional statement and will do so using the following example:

You and your professor discuss your progress in the course and they make the promise to you that, ïf you pass the final, then you will pass the course."The professor's conditional statement can be broken up into the simple statements p : you pass the final, and q : you pass the course.

Now the question is, did your professor keep their promise? Here we will denote the true-false for the simple statements by (p, q).

1. (T,T): You pass the final and you pass the course. The professor did keep their promise, so the conditional statement is true.
2. (T,F): You pass the final, but you fail the course. The professor did not keep their promise, so the conditional statement is false.
3. (F,T): You fail the final and you pass the course. The professor did keep their promise because the promise had the condition that you passed the final. Nothing was said about failing the final, therefore the promise was kept and the statement is true.
4. (F,F): You fail the final and you fail the course. The professor did keep their promise because the promise had the condition that you passed the final. Nothing was said about failing the final, therefore the promise was kept and the statement is true.

Note, the biggest confusion for this truth table are the last two options, however the professor's promise hinged on passing the course. Since that condition was not met, the statement is vacuously true. The truth table can be given as follows:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Question: Create the truth table for $[(p \vee q) \wedge \sim p] \to q$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

This is a tautology and a tautology involving a conditional is called an **implication**.

If we recall how the bicconditional was defined, i.e. that $p \to q$ and $q \to p$ were both true. Then the truth table for $p \leftrightarrow q$ is the same as $(p \to q) \wedge (q \to p)$.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Therefore the truth table for the biconditional is given by

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 8. *You receive a letter in the mail from a company stating the following:*

If your number matches the pre-selected number and you return it by a specified deadline, then you will win $\$1,000,000$.

Now, suppose your number did not match the pre-selected number, you return the letter before the deadline and you only win a free issue of a magazine.

Question: Can you sue the company for false advertisement?
Answer: First break of the conditions for winning into simple statements and create the symbolic statement. This can be done in the following way:

> p : *Your number matches the pre-selected number.*
>
> q : *You return the letter by the deadline.*
>
> r : *You win* $\$1,000,000$.

Looking at the conditions for winning, we see that we can create the following symbolic statement using the above simple statements.

$$(p \wedge q) \rightarrow r.$$

Now, in terms of the simple statements p, q, and r, how does what happen to us look symbolically? Our number did not match the pre-selected number, so our scenario involves p being false. We return the letter by the deadline, so in our scenario we have that q is true. Lastly, we did not win a million dollars, so our scenario has that r is false.

Reminder, we determine true or false by assuming that the p, q, and r are "true," and if our scenario agrees we say true otherwise it disagrees and we say false. To be more precise, if we created a truth table for $(p \wedge q) \rightarrow r$, then we are looking at a specific combination of true and false for the simple statements based on what happens to us. Specifically the combination were p is false, q is true, and r is false. Thus

$$\begin{array}{ll} (p \wedge q) \rightarrow r, & \textit{Replace } p,\ q,\ r \textit{ with correct T or F} \\ (F \wedge T) \rightarrow F, & \textit{Remember } (F \wedge T) \textit{ is F} \\ F \rightarrow F, & \textit{Remember that } F \rightarrow F \textit{ is true} \\ T & \end{array}$$

Since this statement is true for our combination of true and false, then we can conclude that we cannot sue the company.

Note, a false advertisement would have the scenario end up being false.

3.5

In this section we look at equivalent statements and some special variations of the conditional statement.

Definition 9. ***Equivalent statements*** *are made up of the same simple statements and have the same corresponding truth values for each true-false combination.*

If you want to show that two statements are equivalent, the easiest and best way is to produce the truth tables for both. Let us consider the two statements $p \vee \sim q$ and $\sim p \rightarrow \sim q$.

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$\sim p \rightarrow \sim q$
T	T	F	T	F	T
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	T	T	T

For the conditional statement, $p \rightarrow q$, there are three variations of the conditional.

1. Converse: $q \to p$, If q, then p.
2. Inverse: $\sim p \to \sim q$, If not p, then not q.
3. Contrapositive: $\sim q \to \sim p$, If not q, then not p.

The interesting thing that occurs for these variations is that the converse and inverse are equivalent statements, and the contrapositive and the original conditional are equivalent!

Example 10. *Consider the statements p : you live in Atlanta and q : you live in Georgia. Write the conditional, converse, inverse, and contrapositive.*

1. *Conditional: If you live in Atlanta, then you live in Georgia.*
2. *Contrapositive: If you do not live in Georgia, then you do not live in Atlanta.*
3. *Converse: If you live in Georgia, then you live in Atlanta.*
4. *Inverse: If you do not live in Atlanta, then you do not live in Georgia.*

Note that (1) and (2) are both true and (3) and (4) are not necessarily true!

Example 11. *Write the converse, inverse, and contrapositive of the conditional statement: If you are 17, then you are not eligible to vote.*

First thing we need to do is specify our p and q. This can be done by letting p : you are 17 and q : you are eligible to vote. Note that q is not precisely what was stated, but rather written in a way to remove the not. Therefore the statement given can be written as

$$p \to \sim q.$$

1. *Conditional: $p \to \sim q$, If you are 17, then you are not eligible to vote.*

   Note that the following three are based on the above as the conditional statement!

2. *Converse: $\sim q \to p$, If you are not eligible to vote, then you are 17.*
3. *Inverse: $\sim p \to \sim (\sim q)$, If you are not 17, then you are eligible to vote. (double negation is just q!)*
4. *Contrapositive: $\sim (\sim q) \to \sim p$, If you are eligible to vote, then you are not 17.*

The easiest way to remember these is as follows (especially if the symbols are confusing):

1. Converse reverses the antecedent and consequent.
2. Inverse keeps the order of antecedent and consequent of the original conditional, but both are now negated.
3. Contrapositive reverses the antecedent and consequent AND negates both.