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MATH 1001
Quantitative Reasoning

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The purpose of these notes is to provide a concise overview of the material covered in MATH 1001 at Georgia State University. Note that the chapters are presented in order that they are presented in the course due to exam scheduling.

Chapter 2

The main goal of chapter two is to develop notions of sets and their corresponding operations.

2.1

A **set** is a collection of objects, called **elements**, that can be clearly determined. Furthermore, a set must be well-defined, meaning that the elements in the set can be clearly determined. For example, we can consider the set

$$\{\text{NFL quarterbacks }\}$$

which is well-defined because we are able to check roster data to see who is listed as a quarterback. On the other hand,

$$\{\text{Good NFL quarterbacks}\}$$

is not well-defined since we use the subjective word good. What is good? It can differ depending on the person writing out the set.

For sets, we typically represent sets with capital letters and there are three ways we will use to show the contents of our sets. Consider the set $T = \{\text{NFL teams}\}$. The first method is called **word description** and is what we have been using to describe our sets. This is a short description inside $\{\}$ that describes what our elements are. The second method is called **roster method** which involves listing the elements out explicitly. For example,

$$T = \{\text{Bengals, Cardinals, Colts, Broncos, ...}\}$$

and note that in the case of longer rosters we can list enough to show the pattern and use ellipses to continue the list. The last method is the one that we will use the most, called **set-builder notation**. The way to use set-builder is to first pick a general representative, x, y, α, etc, and then have some formulation of what makes x an element of your set. This is similar to the word description sometimes, but as we will see later on it does not have to be. For example,

$$T = \{x | x \text{ is a NFL team}\}.$$

Note, the '|' is used to separate our general element from the formulation of why x is in the set.

One thing we are able to do is convert between the different methods of representing sets. For example, consider the set

$$O = \{x | x \text{ is a positive odd number less than 12}\}$$

which is given in set-builder notation. To find the word description of O, all we need to do is provide a word description of the conditions to the left of '|'. In this case,

$$O = \{\text{ set of positive odd numbers less than 12}\}.$$

The last method is roster, and all we need to do is list the numbers as follows

$$O = \{1, 3, 5, 9, 11\}.$$

In the discussion of sets, there is a special set called the empty set, null-set, denoted $\emptyset$, which represents the set with no elements. It is common to see this set represented in the following way $A = \emptyset$, $\emptyset$, or $\{\}$. In regards to the empty set, we need to be careful. Consider the following two sets

1) $\{x | x \text{ is less than 4 and larger than 5}\}$
2) $\{\emptyset\}$.

Looking at the two sets, 1) is empty since there is no number less than 4 and larger than 5, however 2) is not empty. This is because a set is a collection of objects and those objects can be sets themselves, so $\{\emptyset\}$ is the set containing the empty set and is not empty.

Note: The following are not the same $\emptyset$ and $\{\emptyset\}$. The first is the empty set while the second is the set containing the empty set.

In the case that we want to talk about whether an element is a member of a set, we can use the following symbols.

- The symbol $\in$ indicates that an object is a member of a set.
- The symbol $\notin$ indicates than an object is not a member of a set.

Example 1. *Consider the set E which is the set of positive even numbers and the numbers 6, 7, -2, and 0. Which are members of E?*

$6 \in E$, $7 \notin E$, $-2 \notin E$ *since it is negative, and* $0 \notin E$ *since we are assuming positive numbers and in this class positive means strictly greater than 0.*

In this course, one of the most common sets we will deal with is the set of natural numbers which is the collection of numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. This is especially important when using set-builder notation. For example, the set A of natural numbers less than 6 can be represented as

$$A = \{x | x \in \mathbb{N} \text{ and } x < 6\},$$

or the set B of natural numbers greater than or equal to 10,

$$B = \{x | x \in \mathbb{N} \text{ and } x \geq 10\}.$$

Consider the following ways we can represent inequalities in set-builder notation (these are not all of the possible combinations):

$x < a$	$\{x \| x \in \mathbb{N} \text{ and } x < a\}$
$x \leq a$	$\{x \| x \in \mathbb{N} \text{ and } x \leq a\}$
$x > a$	$\{x \| x \in \mathbb{N} \text{ and } x > a\}$
$x \geq a$	$\{x \| x \in \mathbb{N} \text{ and } x \geq a\}$
$a < x < b$	$\{x \| x \in \mathbb{N} \text{ and } a < x < b\}$
$a \leq x \leq b$	$\{x \| x \in \mathbb{N} \text{ and } a \leq x \leq b\}$

Another question we like to ask in regards to sets is, how many elements are in the set? We answer this using cardinality.

Definition 2. *The **cardinal number** or **cardinality** of a set A is the number of distinct elements in A, denoted by $n(A)$.*

The key to this definition is the word distinct, meaning different. For example if we consider the set $A = \{1, 2, 5, 8\}$, each element in A is different, so $n(A) = 4$. However, the set $B = \{1, 1, 3, 3\}$ has $n(B) = 2$ since we have repeated 1s and 3s (we do not count repeats). Consider the following set

$$C = \{x | x \in \mathbb{N} \text{ and } 10 < x \leq 15\}.$$

We can find the cardinality of the set C by finding the roster for C, which is $C = \{11, 12, 13, 14, 15\}$. Therefore $n(C) = 5$.

Recall the $\emptyset$ from earlier and how it was the only set with no elements, it has cardinality $n(\emptyset) = 0$, and is the only set to do so, i.e. having cardinality zero implies our set is the empty set.

The last concept we want to discuss is as follows. Given two sets A and B, we say that the sets are **equivalent** if they have the same cardinality, i.e. that $n(A) = n(B)$. To be more precise, since A and B have the same cardinality means that we are able to produce a one-to-one correspondence with the elements of A and B.

Definition 3. *Two sets A and B have a **one-to-one correspondence** if each element of A can be paired with exactly one element of B. Note that these correspondence do not have to be unique, we can create multiple ones between two sets.*

Example 4. *Consider the two sets $A = \{x|$ x is a vowel$\}$ and $B = \{1, 2, 3, 4, 5\}$. The first thing we want to do is check for cardinality. Since $A = \{a, e, i, o, u\}$, then $n(A) = n(B) = 5$.*

By our definition, a one-to-one correspondence exists. We can construct one by associating the elements of A and B as follows

$$\begin{aligned} a &\leftrightarrow 1 \\ e &\leftrightarrow 2 \\ i &\leftrightarrow 3 \\ o &\leftrightarrow 4 \\ u &\leftrightarrow 5 \end{aligned}$$

Therefore we showed that A and B are equivalent. Note, that we can produce more one-to-one correspondences by switching the numbers one the right around as well.

The beauty of the one-to-one correspondence is the following

1. If A and B can be placed in a one-to-one correspondence, then $n(A) = n(B)$.
2. If A and B cannot be placed in a one-to-one correspondence, then $n(A) \neq n(B)$.

Example 5 (Application)**.** *The percentage of time devoted to an activity each day for American students is given by 33% spent sleeping, 33% is spent working, 24% is spent at school, and the remaining 10% is spent doing other activities.*

Are the set of activities and the set of percentages in one-to-one correspondence?

Most would say yes, since each activity has its own percentage, however lets consider the set A to be the set of activities and B to be the set of percentages,

$$A = \{\textit{sleep, work, school, other}\} \qquad B = \{33\%, 33\%, 24\%, 10\%\}.$$

If we calculate the cardinalities of A and B we get that $n(A) = 4$ while $n(B) = 3$. The latter occurs because cardinality looks at the set of distinct percentages and we have 33% that appears twice. So when counting the number of distinct percentages, the 33% is only counted once.

Therefore the answer to the question is no, the sets are not in one-to-one correspondence.

We will end this section with a few definitions.

Given a set A, we say that A is **finite** is $n(A) = 0$ or $n(A) \in \mathbb{N}$. A set B is called **infinite** if $n(B) \neq 0$ or $n(B) \notin \mathbb{N}$. Most of the sets we have dealt with are finite, the only exception is the infinite set $\mathbb{N}$.

Lastly, we say two sets are equal if the contain the same elements, which leads to our next section on subsets.

2.2

In this section we will talk about a relationship that can occur between two sets called subsets.

Definition 6. *Given two sets A and B, we say that A is a subset of B, $A \subseteq B$, if every element of A is also an element of B which implies that $n(A) \leq n(B)$. Note that A can equal B since we allow a set to be a subset of itself.*

Let us consider the example $A = \{$set of zodiac signs$\}$ and $B = \{$aries, taurus$\}$. Looking at the two sets, we see that $B \subseteq A$ since aries $\in A$ and taurus $\in A$. However, the reverse is not true since there are several zodiac signs in A that are not in B, i.e. virgo, pisces, etc.

If a set A is not a subset of B, then we use the symbol $A \nsubseteq B$.

Example 7. *Consider the sets $A = \{1, 4, 5, 9\}$ and $B = \{1, 9, 5, 3, 4, 7\}$.*

One can see by checking that $A \subseteq B$ since $1, 4, 5, 9 \in B$, but $B \nsubseteq A$ since $3, 7 \notin A$.

Consider $C = \{a, b, c\}$ and $D = \{c, a, b\}$. One can see that both $C \subseteq D$ and $D \subseteq C$ are both true. This gives rise to different definition.

Definition 8. *For any two sets A and B, if both $A \subseteq B$ and $B \subseteq A$, then the sets A and B are **equal**.*

In the first definition for subset we allowed $n(A) = n(B)$, i.e. the sets $A = B$. However, if we remove that possibility we arrive at the definition of a proper subset.

Definition 9. *Given two sets A and B, we say that A is a proper subset of B, $A \subset B$, if every element in A is also an element of B and $n(A) < n(B)$.*

Example 10. *Consider the set $A = \{1, 4, 5\}$, then one is able to find all of the proper subsets of A by picking all possible combinations of elements of A of size 0,1, and 2.*

Size	*Proper Subsets*
0	$\{\}$
1	$\{1\}, \{4\}, \{5\}$
2	$\{1, 4\}, \{1, 5\}, \{4, 5\}$
3	*There are no proper subsets of size 3.*

Now, how does the $\emptyset$ play into subsets and proper subsets? The easy answer is that for any set B, $\emptyset \subseteq B$, however, we can be more precise as follows:

1. For any set B, $\emptyset \subseteq B$
2. For any set B, with $B \neq \emptyset$, $\emptyset \subset B$

A particular problem in mathematics that we like to do is to count things, in terms of sets and subsets, we would like to know the number of subsets for a particular set. Lets see if we can find a pattern.

Set A	$n(A)$	Subsets of A	Number of Subsets
$\{\}$	0	$\{\}$	1
$\{a\}$	1	$\{\}, \{a\}$	2
$\{a, b\}$	2	$\{\}, \{a\}, \{b\}, \{a, b\}$	4
$\{a, b, c\}$	3	$\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$	8

Do we see a pattern? Using inductive reasoning, one could say that it doubles for every element we add, and that is correct.

For any set of cardinality n, the number of subsets is 2^n. Moreover, since there is only subset that is not proper for any set, we have that the number of proper subsets is $2^n - 1$.
For example, if we consider the set $A = \{x | x \in \mathbb{N} \text{ and } 10 < x \leq 17\}$, we have that $n(A) = 7$ and by the previous statement there are $2^7 = 128$ subsets and $2^7 - 1 = 127$ proper subsets of A.

2.3

In this section we will look at three different set operations, the complement, union, and intersection, and see how they tie into the English language.

For sets it is convenient to consider a general set that contains all the elements under discussion. This set is typically called the **universal set** and is denoted U.

A non-mathematical example could be, consider the set of all MATH 1001 students, then a universal set could be the set of all GSU students taking math. The thing to note is that there can be more than one choice for a universal set since we could have easily said the universal set is the set of all GSU students.

One way we can look at the relationship between two set is through Venn Diagrams and we will consider some examples later.

Definition 11. *We say that two sets A and B are **disjoint** is they have no elements in common.*

Unless it is stated the A and B are disjoint, then there is the possibility that there is some element in common. Remember that some means at least one.

Definition 12. *The **complement of a set** A, denoted A', is the set of all elements in the universal set not in A.*

$$A' = \{x | x \in U \text{ and } x \notin A\}.$$

Example 13. *Consider the universal set* $U = \{a, b, c, d, e, f, g, h\}$ *and the sets* $A = \{a, b, d, e\}$ *and* $B = \{a, b, c, d, e, g, h\}$*. Then the complement of* A *is all of the elements of* U *that are not in* A*, which are* $A' = \{c, f, g, h\}$ *and similarly for* $B' = \{f\}$*.*

The next two operations we will look at uses two sets A and B.

Definition 14. *The* ***intersection*** *of sets* A *and* B*, denoted* $A \cap B$*, is the set of all elements in common for* A *and* B*.*

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

Definition 15. *The* ***union*** *of the sets* A *and* B*, denoted* $A \cup B$*, is the set of all elements in* A *or in* B*.*

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

Example 16. *Consider the sets* $A = \{1, 2, 3, 5, 7, 9, 12\}$ *and* $B = \{1, 2, 3, 9, 12, 13\}$*, then the intersection of* A *and* B *is all of the elements that are both in* A *and* B*,*

$$A \cap B = \{1, 2, 3, 9, 12\}.$$

On the other hand, the union of A *and* B *is all of the elements between the two sets, the only thing that we do is not write the multiplies, i.e. if both* A *and* B *have 1, then we write 1 once. Thus the union is given by*

$$A \cup B = \{1, 2, 3, 5, 7, 9, 12, 13\}.$$

Earlier on in the chapter we talked about the empty-set, how does the intersection and union affect it? Well this question has a nice answer.

1. For any set A, $A \cup \emptyset = A$.
2. For any set A, $A \cap \emptyset = \emptyset$.

The big thing to get from this section is that there is a connection between English and our set operations. Specifically, the following are equivalent, the word or and set union, and the word and with set intersection. Note that these appeared in the set-builder notation for each operations definition.

A common use for this involves the following application which involves the formula for the cardinality of the union of two sets.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

One should note that the we need to subtract $n(A \cap B)$ since the $n(A) + n(B)$ double counts the elements in the interesection.

Example 17 (Application)**.** *A survey of students at Georgia State University asked student whether they would want to donate blood or serve breakfast to the donors. The results are as follows:*

450	*Donate blood*
320	*Serve breakfast to donors*
100	*Donate blood and serve breakfast*

How many students would serve breakfast or donate?

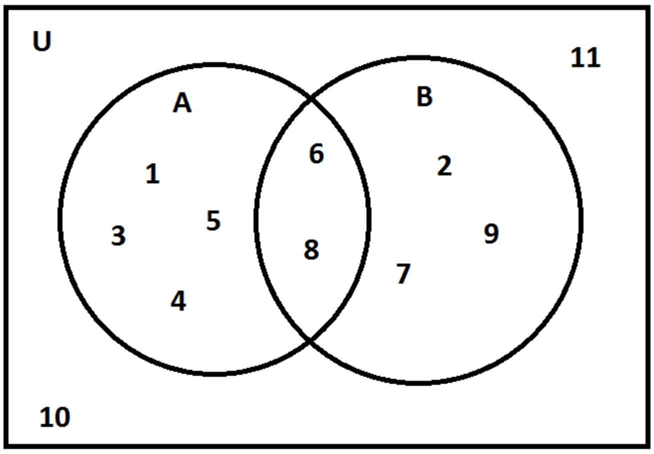
For this survey, we can place the students into sets, i.e. A *be the set of students who would donate blood, and* B *be the set of students who would serve breakfast. Now the remaining group can be represented using the sets* A *and* B*, since the remaining 100 students want to donate AND serve, so they are represented by* $A \cap B$*.*

The other thing to note is that the question asks for the number of students that would serve OR donate, which is $A \cup B$*. Therefore we have*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 450 + 320 - 100 = 670 \text{ students}.$$

A typical question could be, if given three of the following, $n(A)$, $n(B)$, $n(A \cup B)$, $n(A \cap B)$, find the missing value.

Example 18. *Consider the following venn diagram and find the following:*



Note that the universal set is given by $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ *and the sets* A *and* B *are* $A = \{1, 3, 4, 56, 8\}$ *and* $B = \{2, 6, 7, 8, 9\}$.

To find the intersection $A \cap B$*, we need to find the elements shared by both sets, and for a venn diagram that is the part where the circles overlap. Thus*

$$A \cap B = \{6, 8\}.$$

The union $A \cup B$*, is the elements in both* A *and* B*. Thus*

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Note that the elements 10 and 11 are not in the union.

Other typical questions that can bee asked involves complements, unions, and intersections. Consider the following

$$A' \cap B = \{2, 7, 9\}$$

since $A' = \{2, 7, 9, 10, 11\}$ *then the intersection with* $B = \{2, 7, 9, 8, 6\}$ *yields the above.*

$$A \cup B' = \{1, 3, 4, 5, 10, 11\}$$

since $B' = \{1, 3, 4, 5, 10, 11\}$ *and the union is just all the elements in* A *and* B'*. Lastly,*

$$(A \cup B)' = \{10, 11\}$$

since $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and the complement says take everything in the universal set* U *that is not in* $A \cup B$.