MATH 150: Plane Trigonometry **(Summer 2020)**

Overview

Instructor: Joshua Miller

Course Overview:

1 Chapter 1

In chapter one, there are a few big topics. First is unit conversions for angles and there are three main units we are going to to use, $D^\circ M'S''$, degrees, and radians. Depending on the type of problem we are working on, we need to be able to convert between them.

If we want to convert between degrees and radians, it is simple using

$$D^\circ = D \times \frac{\pi}{180} \text{ radians}, \qquad P \text{ radians} = \left(P\frac{180}{\pi}\right)^\circ$$

However, some times we are giving things in $D^\circ M'S''$ and need to be able to convert our information to a more useful unit. That is done as follows

$$D^\circ M'S'' = \left(\frac{3600 \times D + 60 \times M + S}{3600}\right)^\circ.$$

For converting to $D^\circ M'S''$ please see the appropriate discussion post. The other main topic we discuss in the early sections of chapter one has to do with angles with vertices located at the center of circles and the relationship between the sector of the circle subtended by the angle, the arc subtended by the angle, and the radius. For this section we have three important equations based on l, r, A, θ where they are the length of the arc subtended by the angle, radius of the circle, area of the sector subtended by the angle, and the angle's measure, respectively .

$$l = \theta r, \qquad A = \frac{1}{2}lr, \qquad A = \frac{1}{2}\theta r^2.$$

The most important thing is that the units of θ are in radians.

In the next section, we look at a right triangle ΔABC with the right angle located at A. We introduced the basics of the six trigonometric functions, sine, cosine, tangent, and their reciprocals. In this section we learn that the trigonometric values for the angles B and C correspond to ratios of the triangles legs and hypotenuse. Given the angle B of our right triangle, where O stands for the length of the leg opposite of B, A the leg adjacent to B and H, the hypotenuse of our triangle, then the six trigonometric ratios of B are as follows (plus since B and C are complementary angles, then we also have the following since the leg opposite C has been labeled A and therefor the leg adjacent to C is now O):

$$\sin B = \frac{O}{H} = \cos C$$

$$\csc B = \frac{H}{O} = \sec C$$

$$\cos B = \frac{A}{H} = \sin C$$

$$\sec B = \frac{H}{A} = \csc C$$

$$\tan B = \frac{O}{A} = \cot C$$

$$\cot B = \frac{A}{O} = \tan C$$

If you don't see what is happening, draw out the right triangle described with the sides labeled relative to B.

In the last section of chapter one we looked at solving right triangles. For any triangle there are six pieces of information, the three sides and the three angles. However for a right triangle we always know one piece, namely that one angle is $90° = \frac{\pi}{2}$. It turns out that we only need three pieces of information to solve any triangle, thus for a right triangle we must know two other pieces of information. We can either have a side-angle or have a side-side.

Side-Angle: To solve this problem, we use two facts, (1) the the acute angles are complements (so we can find the missing angle) and (2) we can use the trigonometric ratios to find the missing sides. For example, consider the following:

Problem SA: Given a right triangle with right angle at A, and $B = 35°$ and $b = 3$. Solve the triangle.

(Solution) Since B and C are complements, then $B + C = 90$ or in other words

$$C = 90 - B = 90 - 35 = 55°.$$

In this problem we are given the angle B and also the side opposite of B, which is b in this case (called O earlier). So from here, we have a few options, we can solve for the adjacent side to B or the hypotenuse, the only thing that changes is the trigonometric ratio we use. If we want to solve for the adjacent side, we use tangent since its ratio is O/A, hence

$$(1) \qquad \tan B = \frac{O}{A} \qquad \Rightarrow \tan 35 = \frac{3}{A} \qquad \Rightarrow A = \frac{3}{\tan 35}.$$

On the otherhand, we could have solved for the hypotenuse which utilizes sine since its ratio is O/H, hence

$$(2) \qquad \sin B = \frac{O}{H} \qquad \Rightarrow \sin 35 = \frac{3}{H} \qquad \Rightarrow H = \frac{3}{\sin 35}.$$

After finding decimal approximations for A and H we are done. The only other option we have in this problem is either using either (1) or (2) to find a missing side, then use the Pythagorean theorem to find the missing side. This is done as follow

$$(1) \qquad H = \sqrt{O^2 + A^2} \qquad (2) \qquad A = \sqrt{H^2 - O^2}.$$

The other case we consider is when we are given two side lengths.

Side-Side: To solve this problem, we use two facts (1) we can use the ratios and inverse trigonometric functions to find a missing angle and (2) the missing side can be found similarly to the previous side-angle problem. For example, consider the following:

Problem SS: Given a right triangle with right angle at A, and $a = 6$ and $b = 5$. Solve the right triangle.

(Solution) In this problem we are given two pieces of information, the hypotenuse which is a and the side opposite of angle B. We can find the third side using Pythagorean theorem

$$c = \sqrt{a^2 - b^2} = \sqrt{36 - 25} = \sqrt{11}.$$

Next we need to find one of the angle which can be done using which ever trigonometric function you want and either angle B or C. Here we are going to use angle B and the sine function

$$\sin B = \frac{b}{a} = \frac{5}{6} \qquad \Rightarrow B = \arcsin\left(\frac{5}{6}\right).$$

Lastly, we need to find the angle C which can be done using a similar trick to B

$$\cos C = \frac{b}{a} = \frac{5}{6} \qquad \Rightarrow C = \arccos\left(\frac{5}{6}\right),$$

or you can use the fact that $B+C=90°$, so

$$C = 90 - B = 90 - \arcsin\left(\frac{5}{6}\right).$$

Last thing we looked at in this section is angles of elevation, angles from the level line (parallel to ground) up to the line of sight, or angles of depression, angles from the level line down to the line of sight.

Problem Depression: From an observation point 10 meters above ground level, we measure the angle of depression to an object to be 20.5°. Find the distance from the object to the point directly beneath the observation point.

(Solution) From the question, we have two pieces of information, the angle $\alpha = 20.5°$ and the side adjacent to α's complement, which will be called a. So from the problem, we are interested in the side opposite of α's complement called o. Since we have a and need o, the trigonometric function we want to use is tangent.

$$\tan(90° - \alpha) = \frac{o}{a} = \frac{o}{10} \qquad \Rightarrow o = 10\tan\alpha = 10\tan(90° - 20.5°).$$

If the question asked for the distance from the observer to the object, we would want the hypotenuse which can now be found using Pythagorean theorem

$$h = \sqrt{10^2 + (10\tan(90° - 20.5°)} = \sqrt{100 + 100\tan^2(90° - 20.5°)}.$$

Problem Elevation: From an observation point on the ground, the angle of elevation to the top a Joshua tree measures 50°. We move 10 ft away and the angle of elevation now shows 27°. How tall is the tree.

(Solution) In this problem we are going to create two triangles relative to $\alpha = 50°$ and $\beta = 27°$. Here we will label the triangle with α relative to α so $a_\alpha, o_\alpha, h_\alpha$ are the adjacent side, opposite side and hypotenuse. We will label β's triangle similarly with $a_\beta, o_\beta, h_\beta$. Now these two triangles have a side in common, namely o_α and o_β, since they both correspond to the height of the tree and we know that $a_\beta = a_\alpha + 10$ since we moved 10 ft away to find β's measure.
What we gather from this is that we should use tangent since $o = o_\alpha = o_\beta$ and $a_\alpha + 10 = a_\beta$. So

$$\tan\alpha = \frac{o}{a_\alpha} \qquad \tan\beta = \frac{o}{a_\alpha + 10}.$$

Since we want o and the only other unknown is a_α, we will solve the second equation for a_α.

$$\tan\beta = \frac{o}{a_\alpha + 10} \qquad \Rightarrow (a_\alpha + 10)\tan\beta = o \qquad \Rightarrow a_\alpha = \frac{o - 10\tan\beta}{\tan\beta}.$$

This can now be plugged into the first equation.

$$\tan\alpha = \frac{o}{a_\alpha} = \frac{o}{\frac{o - 10\tan\beta}{\tan\beta}} = \frac{o\tan\beta}{o - 10\tan\beta}.$$

All that is needed is to solve for o.

$$\tan\alpha = \frac{o\tan\beta}{o - 10\tan\beta} \qquad \Rightarrow o\tan\alpha - 10\tan\alpha\tan\beta = o\tan\beta \qquad \Rightarrow o\tan\alpha - o\tan\beta = 10\tan\alpha\tan\beta.$$

Thus we solve for o using division and distributive properties.

$$o = \frac{10\tan\alpha\tan\beta}{\tan\alpha - \tan\beta} = \frac{10\tan(50°)\tan(27°)}{\tan(50°) - \tan(27°)}$$

2 Chapter 2

In chapter two we want to start improving on what we did it chapter one and ideally generalize things to a broader set of circumstances. The beginning of chapter two starts off with the basics of vector geometry and some nice formulas. Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the distance between P and Q is given by

$$d(PQ) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Consider the example $P(1, 2)$ and $Q(2, -4)$, then the distance between them is given by

$$d(PQ) = \sqrt{(1-2)^2 + (2-(-4))^2} = \sqrt{1+36} = \sqrt{37}.$$

The other topic discussed early on is about the equation of a circle, usually denoted

$$(x-h)^2 + (y-k)^2 = r^2$$

where the circle is centered at the point (h, k) and has radius r. Now it is always nice when we get an equation in this form since we can immediately get the information we want. However, there are times when we receive equations of the form

$$(1) \qquad ax^2 + bx + ay^2 + cy = d$$

where a, b, c, d are real numbers. Now this doesn't look like our standard equation of a circle, but we can use what is called completing the square to transform it. First step is to divide through by a, this is just a personal preference.

$$x^2 + \frac{b}{a}x + y^2 + \frac{c}{a}y = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + y^2 + \frac{c}{a}y + \frac{c^2}{4a^2} - \frac{c^2}{4a^2} = \frac{d}{a}.$$

It then follows that completing the square is achieved by

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{c}{2a}\right)^2 = \frac{d}{a} + \frac{b^2+c^2}{4a^2}.$$

Therefore (1) is really the equation of the circle with center $(\frac{-b}{2a}, \frac{-c}{2a})$ and radius $r = \sqrt{\frac{d}{a} + \frac{b^2+c^2}{4a^2}}$

Now we will begin looking at vectors, namely that if you take two vectors, $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$ and a real number t,we get a couple of nice facts:

$$\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2] = v + u \qquad \text{and} \qquad t\vec{u} = [tx_1, ty_1].$$

The last nice fact has to deal with magnitude of a vector, a different way to say the length of a vector

$$\text{Magnitude: } \|\vec{u}\| = \sqrt{x_1^2 + y_1^2}, \qquad \|t\vec{u}\| = t\,\|\vec{u}\| \text{ for any real } t.$$

Now that we have the basics of vectors we have a couple of ways to utilize them, first we can take two points, say $P(x, y)$ and $Q(a, b)$ and to find the vector $\vec{PQ}$ where Q is called the terminal point and P is called the initial point, we take terminal-initial in both spaces, i.e.

$$\vec{PQ} = [a-x, b-y].$$

The other tie in for vectors that we want to consider is the tie in to equations of a line, more specifically the general equation of the line

$$ax + by = c$$

where a, b, c are real numbers. The nice thing about this equation is that it tells you everything you need to know about the equation

$$\text{x-intercept: } (c/a, 0) \qquad \text{y-intercept: } (0, c/b) \qquad \text{Direction Vector : } [-b, a].$$

The direction vector is just the slope of the line, using m for slope, which is $m = \frac{-b}{a}$. There is one other form of the line that is beneficial, namely the point-slope form. Given the slope of a line m and a point (h, k) on the line, the point-slope form of the line is given by

$$y - k = m(x-h).$$

The last relevant information for lines is if we have the lines $y = m_1x + b_1$ and $y = m_2x + b_2$, then the lines are parallel if $m_1 = m_2$ or perpendicular if $m_1 = \frac{-1}{m_2}$. Let's consider an example now.

Problem Lines: Given the line $L := 3x + 4y = 2$ find the equations of line perpendicular and parallel to L through the point $(1, 1)$.
(Solution) For the perpendicular equation, the general equation is

$$-4x + 3y = c \qquad \Rightarrow -4(1) + 3(1) = -1 = c.$$

Thus the equation is $-4x + 3y = -1$, the thing to learn from this is that the perpendicular direction vector is $[a.b]$ so we just flip the a, b on the original and negate one of them. For the parallel equation it is similar,

$$3x + 4y = c \qquad \Rightarrow 3(1) + 4(1) = 7 = c.$$

Thus the parallel line is $3x + 4y = 7$.

Prior to this chapter, we had no standard way to talk about angles until we introduced the concept of standard rotational angles. If we recall we used two rays, the initial and terminal rays, to define the angle. Now we can standardize talking about angles by requiring that the initial ray coincide with the positive x-axis. With this now set, we say counter-clockwise rotation coincides with positive angle measure and clock-wise rotation coincides with negative angle measure.

There are a few things we need to discuss.

(Question) How do we know if two angles are equivalent, ie coterminal?
We say two angles are coterminal if the initial sides coincide and the rotational measures differ by an integer multiple of one turn, ($360°$ or 2π). For example, the angles $-315° = 45°, 405°$, and $765°$ are all coterminal since they differ by integer multiples of $360°$.

(Question) What is the relationship between angles and the coordinate system?
If we consider the point $P(1, 0)$ we can take some rotational measure α and α-rotate $P(1, 0)$ about the origin, and it turns out that resulting point $P_\alpha(x, y)$ has coordinates given by

$$x = \cos\alpha \qquad y = \sin\alpha.$$

We can generalize this further to α-rotating $P(r, 0)$ for $r \geq 0$ by letting

$$x = r\cos\alpha \qquad y = r\sin\alpha \qquad \text{where } r = \sqrt{x^2 + y^2}.$$

What this now allows us to do is tell us what quadrant our point, and thus our terminal ray, is in based on the sign of $\cos\alpha$ and $\sin\alpha$.

Problem Find Trig Values: Given α in standard position in the third quadrant with $\cos\alpha = -\dfrac{\sqrt{3}}{2}$, find the other trigonometric functions of α.

(Solution) The fact that we know α is in the third quadrant tells us that $\cos\alpha < 0$ and $\sin\alpha < 0$ which means x and y are both negative. Thus it follows

$$\cos\alpha = \frac{-\sqrt{3}}{2} = \frac{x}{r} \qquad \Rightarrow x = -\sqrt{3} \text{ and } r = 2.$$

Thus we only need to solve for y which is done by Pythagorean theorem, note that we are in quadrant three so y is negative

$$y = -\sqrt{r^2 - x^2} = -\sqrt{2^2 - (-\sqrt{3})^2} = -\sqrt{4 - 3} = \sqrt{1} = -1.$$

Hence $\sin\alpha = \dfrac{-1}{2}$, $\tan\alpha = \dfrac{1}{\sqrt{3}}$, and cosecant, secant, and cotangent are found by taking the appropriate reciprocal.

Another topic is that we want to find values of angles that are larger than 360° or smaller than −360°, and that is accomplished in a rather neat way. If we take one of these angles, say α and rewrite it as

$$\alpha = n \times 360° + r \qquad \text{where n is an integer and } 0 < r < 360°.$$

It then follows that

$$\sin\alpha = \sin r, \qquad \cos\alpha = \cos r, \qquad \tan\alpha = \tan r,$$

and the reciprocal functions follow as well. Note this works if we are in radians as well, we just replace 360° with 2π.

Problem Periodicity Problems: Compute $\sin 1125°$.

(Solution) We see that 1125° is not in our normal range, so we want to rewrite it. This is done using division. Since $3 \times 360 = 1080$, then it follows that $1125° = 3 \times 260° + 45°$. Thus $\sin 1125° = \sin 45° = \frac{\sqrt{2}}{2}$.

Besides these topics, there are some key points that we want to consider that will be combined for ease of view. Let us now consider the three main functions, sine, cosine, and tangent:

	Sine	Cosine	Tangent
Domain:	All Reals	All Reals	All Reals except $\frac{\pi}{2}$ multiples
Range:	[-1,1]	[-1,1]	All Reals
Y-Intercept:	0	1	0
X-Intercept:	$k\pi$	$\frac{\pi}{2} + k\pi$	$k\pi$
Max:	1 at $\frac{\pi}{2} + 2k\pi$	1 at $2k\pi$	∞
Min:	-1 at $\frac{-\pi}{2} + 2k\pi$	-1 at $(2k+1)\pi$	$-\infty$
Increasing:	$\left[\frac{-\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right]$	$[-\pi + 2k\pi, 2k\pi]$	$\left(\frac{-\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$
Decreasing:	$\left[\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right]$	$[2k\pi, \pi + 2k\pi]$	Never
Even/Odd:	Odd	Even	Odd
Period:	2π	2π	π

Now we can look at solving equations using some clean methods, consider

$$\sin\theta = x,$$

so provided $-1 \leq x \leq 1$ then the main solution is an angle θ such that $-\pi/2 \leq \theta \leq \pi/2$. Moreover, since trigonometric functions are periodic, then all solutions are of the form

$$\sin\theta = x \qquad \theta_k = (-1)^k \arcsin x + k\pi.$$

This can be done similarly for cosine,

$$\cos\theta = x \qquad \theta_k = \pm \arccos x + 2k\pi,$$

however the main θ is an angle satisfying $0 \leq \theta \leq \pi$. Lastly, we can deal with tangent, but it is a little different from the others,

$$\tan\theta = x \qquad \theta_k = \arctan x + k\pi,$$

however x can be any real number and the main θ satisfies $-\pi/2 < \theta < \pi/2$.

Problem Solving Sine: Solve $\sin\theta = \frac{\sqrt{3}}{2}$.

(Solution) The power of these arc-functions is that the answers are simply applications of the the formulas above. After checking that $\sqrt{3}/2 \leq 1$, we can apply the formula

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad \theta_k = (-1)^k \arcsin\frac{\sqrt{3}}{2} + k\pi = (-1)^k\frac{\pi}{3} + k\pi.$$

This gives us all solutions, and the main solution solution corresponds to $k = 0$ which is $\pi/3$ in this case.

Problem Solving Secant: Solve $\csc\theta = 2$.

(Solution) Solving cosine and tangent are similar to sine, but it is a little different if we have a reciprocal. Solving $\csc\theta = 2$ is equivalent to solving $\cos\theta = 1/2$ which we have formulas for, namely

$$\theta_k = \pm\arccos\frac{1}{2} + 2k\pi = \pm\frac{\pi}{3} + 2k\pi.$$

Like previously, the main solution corresponds to $k = 0$ and taking the (+) since 1/2¿0, thus $\theta = \dfrac{\pi}{3}$ is the main solution. Equations involving tangent or cotangent are done similarly.

The second to the last topic we need to discuss from chapter two is the shifted sinusoidal graphs, these are graphs of the form

$$f(x) = a\sin(bx + c) \qquad \text{where a,b,c are real number.}$$

For these two functions, there are two important pieces of information, the period and the phase shift, these are calculated as follow

$$\text{Period}=\frac{2\pi}{b} \qquad \text{Phase Shift, } \phi = \frac{-c}{b}.$$

The thing to note is that for the non-shifted sine graph, there is no phase-shift. Now lets view the properties for the shift sine wave and the non-shifted sine wave (c=0) and corresponds to column one:

	$a\sin bx$	$a\sin(bx+c)$
Domain:	All Reals	All Reals
Range:	$[-a, a]$	$[-a, a]$
Amplitude:	a	a
Y-Intercept:	0	$\sin c$
X-Intercept:	$\frac{n\pi}{b}$	$\phi + \frac{n\pi}{b}$
Maximum:	a at $x = \frac{\pi}{2b} + \frac{2n\pi}{b}$	a at $x = \phi + \frac{\pi}{2b} + \frac{2n\pi}{b}$
Minimum:	-a at $x = \frac{3\pi}{2b} + \frac{2n\pi}{b}$	-a at $x = \phi - \frac{\pi}{2b} + \frac{2n\pi}{b}$
Increasing:	$\left[\frac{\pi}{2b} + \frac{2n\pi}{b}, \frac{\pi}{2b} + \frac{2n\pi}{b}\right]$	$\left[\phi - \frac{\pi}{2b} + \frac{2n\pi}{b}, \phi + \frac{\pi}{2b} + \frac{2n\pi}{b}\right]$
Decreasing:	$\left[\frac{\pi}{2b} + \frac{2n\pi}{b}, \frac{3\pi}{2b} + \frac{2n\pi}{b}\right]$	$\left[\phi + \frac{\pi}{2b} + \frac{2n\pi}{b}, \phi + \frac{3\pi}{2b} + \frac{2n\pi}{b}\right]$
Period:	$\frac{2\pi}{b}$	$\frac{2\pi}{b}$

Now lets tackle a problem, solve a non-shifted is similar to the shifted, so we will focus on the shifted version.

Problem Find the information related to $f(x) = 3\cos(\pi x - \pi/6)$.

(Solution) Now the first thing you are thinking is, this is cosine and we haven't dealt with any of that this section. You are partially right, however we have a formula that says

$$\cos\alpha = \sin(\pi/2 - \alpha).$$

Therefore we can translate our problem into a shift-sine problem by letting $\alpha = \pi x - \pi/6$. Now consider the new equivalent function

$$f(x) = 3\cos(\pi x - \pi/6) = 3\sin(\pi/2 - (\pi x - \pi/6)) = 3\sin(2\pi/3 - \pi x).$$

Looking at the previous chart we need to find the phase-shift and then it is all formulas, here $a = 3$, $b = -\pi$, and $c = 2\pi/3$. Thus

$$\phi = -\frac{c}{b} = -\frac{2\pi/3}{-\pi} = \frac{2}{3}.$$

	$a\sin(bx+c)$
Domain:	All Reals
Range:	$[-3, 3]$
Amplitude:	3
Y-Intercept:	$\sin 2\pi/3 = \sqrt{3}/2$
X-Intercept:	$\frac{2}{3} - \frac{n\pi}{\pi}$
Maximum:	3 at $x = \frac{2}{3} - \frac{1}{2} - 2n$
Minimum:	-3 at $x = \frac{2}{3} + \frac{1}{2} - 2n$
Increasing:	$\left[\frac{2}{3} + \frac{1}{2} - 2n, \frac{2}{3} - \frac{1}{2} - 2n\right]$
Decreasing:	$\left[\frac{2}{3} - \frac{1}{2} - 2n, \frac{2}{3} - \frac{3}{2} - 2n\right]$
Period:	2, taking absolute value here.

The whole point of this chapter is to ultimately be able to apply everything we learned into one problem. Consider the following

Problem Find all solutions to $2\sin(3x - \pi/6) = 1$

(Solution) First thing we want to do is simplify as much as possible to make it look like $\sin a = \#$ and that is accomplished by division and substitution

$$2\sin(3x - \pi/6) = 1 \qquad \Rightarrow \qquad \sin(3x - \pi/6) = 1/2.$$

Now we can use substitution, let $a = 3x - \pi/6$, and thus we now have

$$\sin a = 1/2$$

which has solutions $a = (-1)^k \arcsin(1/2) + \pi k = (-1)^k \frac{\pi}{6} + \pi k$. Now we know that $a = 3x - \pi/6$, thus back-substitution yields

$$3x - \frac{\pi}{6} = (-1)^k \frac{\pi}{6} + \pi k \qquad \Rightarrow \qquad x_k = \frac{\pi}{18} + (-1)^k \frac{\pi}{18} + \frac{\pi k}{3}.$$

Problem Find all solutions to $\cos^2 w + 3\sin w = 3$.

(Solution) First think we want to do is get everything into one function, that is achieved using $\cos^2 w = 1 - \sin^2 w$. So we are really solving

$$1 - \sin^2 w + 3\sin w = 3 \qquad \Rightarrow \qquad \sin^2 w - 3\sin w + 2 = 0$$

which is a quadratic equation under the substitution $t = \sin w$. It follows that

$$t^2 - 3t + 2 = 0 \qquad \Rightarrow \qquad t = \frac{3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm 1}{2} \qquad \rightarrow \qquad t = 2, 1.$$

Now $\sin w = 2$ doesn't work, so the only solution is

$$\sin w = 1 \qquad \Rightarrow \qquad w_k = (-1)^k \arcsin(1) + \pi k = (-1)^k \frac{\pi}{2} + \pi k.$$

3 Chapter 3

In chapter 3 we revisit vectors, by tying the concepts to standard position angles. First we introduce two new products, the dot product and skew product. Given the vectors $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$, the dot product is given by

$$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 = \vec{v} \cdot \vec{u}.$$

Side note, if two vectors have a zero dot product, then the vectors are parallel. The other product is called the skew product,unlike the dot product, order does matter, which is given by

$$\vec{u} \wedge \vec{v} = x_1 y_2 - y_1 x_2 = -(\vec{v} \wedge \vec{u}).$$

The nice fact that we want to build up to is that the vector norms and dot products have a relation, namely

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$; $0 \leq \theta \leq \pi$. What this allows us to do now is for any two vectors, we can find the angle between them by

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right).$$

Consider the following example:

Problem Vector Cosine: Given three points $A(1, 2)$, $B(2, 3)$, and $C(-2, -1)$ find the $\measuredangle BAC$.

(Solution) First thing to note is that we want the angle labeled BAC which means we want to find the vectors $\vec{AB}$ and $\vec{AC}$, so remember terminal minus initial.

$$\vec{AB} = [2 - 1, 3 - 2] = [1, 1] \qquad \vec{AC} = [-2 - 2, -1 - 3] = [-4, -4].$$

The other pieces of information we need to find is the magnitudes and dot product.

$$\left\|\vec{AB}\right\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \qquad \left\|\vec{AC}\right\| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32}$$

$$\vec{AB} \cdot \vec{AC} = (1)(-4) + (1)(-4) = -8.$$

Thus we can now find θ as follow

$$\theta = \arccos\left(\frac{-8}{\sqrt{2}\sqrt{32}}\right).$$

The next question we want to ask is there a relation between skew product and norms? The answer is yes, and takes the form

$$|\vec{u} \wedge \vec{v}| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$$

where θ is again the angle between the two vectors and $0 \leq \theta \leq \pi$. This gives us another way to calculate the angle between the two vectors, ie

$$\theta = \arcsin\left(\frac{|\vec{u} \wedge \vec{v}|}{\|\vec{u}\| \, \|\vec{v}\|}\right).$$

In the words of Billy, but wait, there's more. Using vector arithmetic we can get the area of the triangle that is enclosed by points A, B, and C as follow :

$$\text{area } (\Delta ABC) = \frac{1}{2}\left|\vec{AB} \wedge \vec{AC}\right|$$

Moreover, if we utilize vector addition we also get the area of the resulting parallelogram relative to $\vec{AB}$ and $\vec{AC}$ as

$$\text{area(Parallelogram)} \;= \left|\vec{AB} \wedge \vec{AC}\right|.$$

Let us not consider the following example:

Problem Vector Sine: Given three points $A(1, 2)$, $B(2, 3)$, and $C(-2, -1)$ find the $\measuredangle BAC$, the area of the triangle ΔABC and the area of the resulting parallelogram.

(Solution) We want to find the vectors $\vec{AB}$ and $\vec{AC}$, so remember terminal minus initial.

$$\vec{AB} = [2-1, 3-2] = [1,1] \qquad \vec{AC} = [-2-2, -1-3] = [-4,-4].$$

The other pieces of information we need to find is the magnitudes and skew product.

$$\left\|\vec{AB}\right\| = \sqrt{(1)^2+(1)^2} = \sqrt{2} \qquad \left\|\vec{AC}\right\| = \sqrt{(-4)^2+(-4)^2} = \sqrt{32}$$

$$\vec{AB} \wedge \vec{AC} = (1)(-4) - (1)(-4) = 0.$$

Therefore we can find area as follows:

$$\text{area(Parallelogram)} = |-8| = 8 \qquad \text{area}(\Delta ABC) = \frac{1}{2}\,|-8| = 4.$$

A very important result we get from vectors is called the Generalized Pythagorean Theorem, which takes two forms:

Theorem 1 *Given vectors $\vec{u}$ and $\vec{v}$ the following holds:*

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}\cdot\vec{v})$$
$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v})$$

These two formulas are highly important, especially the first one, since when we consider the case of a triangle ΔABC with vectors $\vec{AB}$ and $\vec{AC}$ like we did earlier, we can reformulate the theorem to solve special cases of triangles. If you recall, a triangle has six pieces of information, three side lengths and three angle measures. It turns out that to solve any triangle, we need to know any three of six, and in most cases this will give us the unique triangle relative to the pieces we are given. We will consider the type of triangles we can solve, in cases, but first lets build up a piece of machinery first. It is with the help of the Generalized Pythagorean theorem that we are able to derive the Law of Cosines which comes in several forms:

Theorem 2 *Given a triangle with sides a, b, and c with corresponding angles, A, B, and C, opposite the respective side, then the following hold:*

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 + 2ac\cos B$$
$$c^2 = a^2 + b^2 + 2ab\cos C$$

Which gives the following three inequalities

$$a \leq b+c, \qquad b \leq a+c, \qquad c \leq a+b$$

This formulation allows us to calculate side lengths by taking the square root. Another formulation that allows us to calculate the angle measures is as follow:

Theorem 3 *Given a triangle with sides a, b, and c with corresponding angles, A, B, and C, opposite the respective side, then the following hold:*

$$A = \arccos\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$B = \arccos\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$C = \arccos\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

Now, the only thing you need to know when using this formulation is that the calculator needs to be set to the correct mode depending on the context of the problem.

Now that we have the law of cosines, we can solve two different families of triangles. The first family is called the Side-Side-Side (SSS) triangle where all three side lengths are known. Therefore to solve the triangle, we need to find the angle measures.

Solving SSS:

1. Find the angle corresponding to the largest side.
2. Find any of the other two angles, then use the fact that the sum of the angles of a triangle add to 180° or $\pi/2$

Problem SSS Triangle: Solve the triangle corresponding to side lengths $a = 5$, $b = 10$, and $c = 13$
(Solution) The nice thing about these is that it is a direct application of our formulas. Looking at the angles, we see that c is the largest side, thus we solve for angle C

$$C = \arccos\left(\frac{b^2 + c^2 - a^2}{2ab}\right) = \arccos\left(\frac{5^2 + 10^2 - 13^2}{2(5)(10)}\right) = \arccos\left(\frac{-19}{100}\right).$$

We can then find the next angle, say B in a similar manner,

$$B = \arccos\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \arccos\left(\frac{5^2 + 13^2 - 10^2}{2(5)(13)}\right) = \arccos\left(\frac{94}{130}\right).$$

Lastly we can find A as follows

$$A = 180 - B - C = 180 - \arccos\left(\frac{-19}{100}\right) - \arccos\left(\frac{94}{130}\right).$$

Note, I am picking to do my calculation in degrees, this can be done in radians by setting the calculator to the correct mode and using $\pi/2$ instead of 180°.

The next triangle case that utilizes law of cosines is the Side-Angle-Side case, meaning that you are given two sides and the angle in between them. These can take the form of

$$(1)\ c, A, b \qquad (2)\ b, C, a \qquad (3)\ a, B, c$$

where the information given matches one of the three cases. You should note that all three cases above are the same, the only difference is names of the given information.

Solving SAS:

1. Find the missing side using law of cosines
2. Find one of the missing angles
3. Find the third angle using subtraction

Problem SAS Triangle: Solve the triangle corresponding to $a = 5$, $C = 45°$, and $b = 6$.
(Solution) The first thing we want to do is find the missing third side, in this case we are looking for side c. This is accomplished by

$$c = \sqrt{b^2 + a^2 - 2ab\cos C} = \sqrt{6^2 + 5^2 - 2(5)(6)\cos 45} = \sqrt{25 + 36 - 60\frac{\sqrt{2}}{2}} = \sqrt{61 - 30\sqrt{2}}.$$

Now that we have the third side, we can use the second version of law of cosines to find an angle, here we will find B.

$$B = \arccos\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \arccos\left(\frac{5^2 + (\sqrt{61 - 30\sqrt{2}})^2 - 6^2}{2(5)\sqrt{61 - 30\sqrt{2}}}\right) = \arccos\left(\frac{50 - 30\sqrt{2}}{10\sqrt{61 - 30\sqrt{2}}}\right).$$

Now that we have B, the measure of A follows

$$A = 180 - B - C = 180 - \arccos\left(\frac{50 - 30\sqrt{2}}{10\sqrt{61 - 30\sqrt{2}}}\right) - 45.$$

Now note, these are all exact answers, if we take approximations they are no longer exact anymore and depending on the amount of decimals we take, the accuracy can be off. The SSS and SAS cases are the only two that can be solved using law of cosines. To be able to solve the last three cases, we need another tool. This time around it will use sine, consider now the law of sines.

Theorem 4 *Given a triangle with sides a, b, and c with corresponding angles, A, B, and C, opposite the respective side, then the following hold:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The next two cases we will considered are solved in a similar manner, they are the Angle-Side-Angle (ASA) and the Angle-Angle-Side (AAS) case.

Solving ASA/AAS

1. Find the third angle by subtracting the other two from 180°
2. The missing sides can be found using the law of sine ratios

The only difference between the two cases is that for the ASA case the side given is not opposite any of the given angles while the AAS case has the given side opposite of one of the given angles.

Problem ASA Triangle: Solve the triangle corresponding to $a = 5$, $B = 50°$ and $C = 42°$.
(Solution) First thing to note is that the side given, a, is not opposite any of the given angles, so this is an ASA case. Now, we use subtraction to find the measure of angle A,

$$A = 180 - B - C = 180 - 50 - 42 = 88°.$$

After that calculation, we now have both parts of the ratio that corresponds to A, namely a and we can calculate $\sin A$, so we will use it to find the missing sides. So by law of sines we have the following

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \Rightarrow b = \frac{a \sin B}{\sin A} = \frac{5 \sin 50}{\sin 88}$$

which is the exact value of b. It follows by a similar argument that we can find c as follows

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \Rightarrow c = \frac{a \sin C}{\sin A} = \frac{5 \sin 42}{\sin 88},$$

again this is the exact value of c. Depending on the context of the problem, we might need a decimal approximation which is done by way of a calculator set to degrees.

Problem AAS Triangle: Solve the triangle corresponding to $a = 5$, $B = 50°$ and $A = 42°$.

(Solution) We see that this is an AAS triangle since we have both side a and angle A. Now, we can find C using subtraction

$$C = 180 - A - B = 180 - 50 - 42 = 88°.$$

Now that we have the last angle measures, sides b and c can be found. So by law of sines we have the following

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \Rightarrow b = \frac{a \sin B}{\sin A} = \frac{5 \sin 50}{\sin 42}$$

which is the exact value of b. It follows by a similar argument that we can find c as follows

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \Rightarrow c = \frac{a \sin C}{\sin A} = \frac{5 \sin 88}{\sin 42},$$

again this is the exact value of c.

Before we can tackle the last case, we need to look at the side-ordering principle which state simply says:

Side-Ordering: If $a < b$ then the angle measures obey $A < B$.

Now that we have the side-ordering property we can attack the Side-Side-Triangle which happens when we are given two side lengths and an angle that corresponds to one of the sides,

$$(1) a, b, A \quad (2) a, b, B \quad (3) a, c, A \quad (4) a, c, C \quad (5) b, c, B \quad (6) b, c, C$$

Solving SSA:

1. Find the angle corresponding to the other side that is given. Note, we can have no solution by the side-ordering principle.
2. Next check both solutions obey the side-ordering principle.
3. Solve the triangle(s).

Problem SSA No Solution: Solve the triangle given $a = 11$, $b = 7$ and $B = 91°$.

(Solution) Looking at the sides given, $a > b$ so by the side-ordering property we must have $A > B$, but B is an obtuse angle. This means A is also obtuse, which cannot happen. Thus no triangle exists.

Problem SSA Two Solutions: Solve the triangle given $a = 10$, $b = 8$, and $B = 40°$.

(Solution) By the side-ordering principle we have $a > b$ so $A > B$. Now we use law of sines to find the measure of A as follows

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \Rightarrow \qquad \sin A = \frac{a \sin B}{b} \qquad \Rightarrow \qquad A = \arcsin\left(\frac{a \sin B}{b}\right).$$

Now we need to remember that the last part has two solutions to it, namely

$$A_1 = \arcsin\left(\frac{10 \sin 40}{8}\right) \cong 53.46° \qquad A_2 = 180° - \arcsin\left(\frac{10 \sin 40}{8}\right) \cong 126.53°.$$

Based on the approximates, both A_1 and A_2 are valid so we need to find two triangles.

Triangle 1, taking A_1, we find C_1 using subtraction and c_1 using our law of sines ratios.

$$C_1 = 180 - B - A_1 = 86.535° \qquad c_1 = \frac{b \sin C_1}{\sin B} = \frac{8 \sin 86.535°}{\sin 40°} \cong 12.423.$$

Triangle 2, taking A_2, we find C_1 using subtraction and c_2 using our law of sines ratios.

$$C_2 = 180 - B - A_2 = 13.464° \qquad c_2 = \frac{b \sin C_2}{\sin B} = \frac{8 \sin 13.464°}{\sin 40°} \cong 2.897.$$

Polar Plane:

Nice thing about dealing, with angles in standard position is that for any real numbers, x and y, there exists an angle θ such that $P(x, y)$ is on the terminal side of our angle. Furthermore, we have that

$$x = (\sqrt{x^2 + y^2}) \cos\theta \qquad y = (\sqrt{x^2 + y^2}) \sin\theta \qquad \text{where } -\pi < \theta \leq \pi.$$

Lastly, the angle θ is defined in the following way

$$\theta = (\text{sign of } y) \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right),$$

the only other convention we take is that if $y = 0$, then the sign of y is positive. Note, the denominator should look familiar, that is just r, which is the distance from P to the origin or the magnitude of the vector $[x, y]$.

With this information in hand, we can now define the Polar Coordinate System. For a point $P(x, y)$, the polar coordinate representation of P is the ordered pair (r, θ) satisfying

$$x = (\sqrt{x^2 + y^2}) \cos\theta \qquad y = (\sqrt{x^2 + y^2}) \sin\theta \qquad \text{where } -\pi < \theta \leq \pi.$$

$$\theta = (\text{sign of } y) \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right), \qquad r = \pm\sqrt{x^2 + y^2}.$$

The big note here is that we are allowing r to be negative. This allows us to have multiple representations, (r, θ) with $r \geq 0$ and θ satisfying the above is call the standard polar representation, but all representations take the form

$$(r, \theta) = (r, \theta + 2n\pi) = (-r, \theta + \pi + 2n\pi).$$

The steps and formulas above give us a way to convert from Polar to Rectangular form, so lets look at Rectangular to Polar conversion.

To Polar Conversion:

1. If $x = y = 0$, then let $r = 0$ and let θ be an real number.
2. If either x or y is non-zero, then the standard polar representation is given by $\left(\sqrt{x^2 + y^2}, (\text{sign of y}) \arccos(x/r)\right)$.

Problem Find the polar form of $P(-1, \sqrt{2})$ and the rectangular form of $Z(2, \pi/4)$.

(Solution) Taking Z to rectangular form is simple since $r = 2$ and $\theta = \pi/4$, thus

$$Z\left(2, \frac{\pi}{4}\right) = (r\cos\theta, r\sin\theta) = \left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = \left(2 \times \frac{\sqrt{2}}{2}, 2 \times \frac{\sqrt{2}}{2}\right) = (\sqrt{2}, \sqrt{2}).$$

For P, we need to find r and θ which can be found using the formulas from earlier

$$r = \pm\sqrt{(-1)^2 + (\sqrt{2})^2} = \pm\sqrt{3} \qquad \theta_0 = +\arccos\left(\frac{-1}{\sqrt{3}}\right) \cong 2.1897.$$

Thus the standard form is $(\sqrt{3}, \theta_0)$ and all forms are of the form $(\sqrt{3}, \theta_0 + 2k\pi) = (-\sqrt{3} + \theta_0 + \pi + 2k\pi)$.

Another nice thing about polar form is that the equations we used for x, y, and r will allow us to translate rectangular equations into polar equations using appropriate substitutions. We will show that using two examples.

Problem Conversion: Convert $3x - 5y = 4$ to its polar form.

(Solution) Simply put, we use the substitutions $x = r\cos\theta$ and $y = r\sin\theta$. Hence

$$3x - 5y = 4 \qquad \Rightarrow \qquad 3r\cos\theta - 5r\sin\theta = 4.$$

Now we will solve for r if we are able to.

$$3r\cos\theta - 5r\sin\theta = 4 \qquad \Rightarrow \qquad r(3\cos\theta - 5\sin\theta) = 4 \qquad \Rightarrow \qquad r = \frac{4}{3\cos\theta - 5\sin\theta}.$$

Problem Conversion: Convert $8\cos\theta + 3\sin\theta = r$.

(Solution) Simply put, we once again use our substitutions $\cos\theta = x/r$ and $\sin\theta = y/r$ which gives us

$$8\cos\theta + 3\sin\theta = r \qquad \Rightarrow \qquad 8\frac{x}{r} + 3\frac{y}{r} = r.$$

We get rid of the r's in the denominator and use the fact that $r^2 = x^2 + y^2$. Thus

$$8\frac{x}{r} + 3\frac{y}{r} = r \qquad \Rightarrow \qquad 8x + 3y = r^2 \qquad \Rightarrow \qquad 8x + 3y = x^2 + y^2.$$

Therefore $0 = x^2 - 8x + y^2 - 3y$ is the answer.

4 Chapter 4

The main machinery that is working behind the scenes in chapter fours is that

$$\cos(\alpha-\beta)=\vec{u_\beta}\cdot\vec{u_\alpha} \qquad \sin(\alpha-\beta)=\vec{u_\beta}\wedge\vec{u_\alpha}$$

where $\vec{u_\beta}=[\cos\beta,\sin\beta]$ and $\vec{u_\alpha}=[\cos\alpha,\sin\alpha]$. Using this machinery we get the Angle Sum/Difference formulas:

Theorem 5 *Given angles α and β then following hold*

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta.$$

The nice thing is that everything is done in terms of sine and cosine for both angles. Lets consider an example.

Problem Sum/Difference Angles: Suppose α is in QIII and $\tan\alpha=\frac{4}{3}$ and β is in QII and $\sin\beta=\frac{5}{13}$. Find sine and cosine of $\gamma=\alpha-\beta$ and $\delta=\alpha+\beta$.

(Solution) What we need to do is find $\sin\alpha$, $\cos\alpha$, and $\sin\beta$ to complete our calculations. Luckily this is something we have done before.

$$\tan\alpha=\frac{4}{3} \quad\Rightarrow\quad x_\alpha=-3 \text{ and } y_\alpha=-4 \quad\Rightarrow\quad r_\alpha=\sqrt{(-3)^2+(-4)^2}=5$$

Thus $\sin\alpha=\frac{-4}{5}$ and $\cos\alpha=\frac{-3}{5}$. It follows from a similar method for β

$$\sin\beta=\frac{5}{13} \quad\Rightarrow\quad r_\beta=13 \text{ and } y_\beta=5 \quad\Rightarrow\quad x_\beta=-\sqrt{13^2-5^2}=-12$$

that $\cos\beta=\frac{-12}{13}$.It then follows

1. $\cos(\alpha+\beta)=\frac{-3}{5}\times\frac{-12}{13}-\frac{-4}{5}\times\frac{5}{13}=\frac{36+20}{65}=\frac{56}{65}$
2. $\cos(\alpha-\beta)=\frac{-3}{5}\times\frac{-12}{13}+\frac{-4}{5}\times\frac{5}{13}=\frac{36-20}{65}=\frac{16}{65}$
3. $\sin(\alpha+\beta)=\frac{-4}{5}\times\frac{-12}{13}+\frac{-3}{5}\times\frac{5}{13}=\frac{48-15}{65}=\frac{33}{65}$
4. $\sin(\alpha-\beta)=\frac{-4}{5}\times\frac{-12}{13}-\frac{-3}{5}\times\frac{5}{13}=\frac{48+15}{65}=\frac{63}{65}$

Another application we typically see with these formula are in cases similar to the following

Problem Sum/Differences with Arc Functions: Find $\sin\psi$ and $\cos\psi$ given that $\psi=\arcsin\frac{12}{13}-\arccos\frac{6}{7}$.

(Solution) It may seem like this is a totally different problem from the previous, but it is not. In actuality it is a lot easier since we have some nice formulas. First let

$$\alpha=\arcsin\frac{12}{13} \text{ and } \beta=\arccos\frac{6}{7},$$

then we get two things for free,

$$\sin\alpha=\sin(\arcsin(12/13))=12/13 \text{ and } \cos\beta=\cos(\arccos(6/7))=6/7.$$

We can now use the formulas

$$\cos(\arcsin x)=\pm\sqrt{1-x^2} \qquad \sin(\arccos x)=\pm\sqrt{1-x^2}.$$

Thus the last pieces we need are applications of these two formulas

$$= \cos\alpha = \cos(\arcsin 12/13) = \sqrt{1-(12/13)^2} = \frac{5}{13} \qquad \sin\beta = \sin(\arccos 6/7) = \sqrt{1-(6/7)^2} = \frac{\sqrt{13}}{7}.$$

$$\sin\psi = \frac{12}{13}\times\frac{6}{7} - \frac{5}{13}\times\frac{\sqrt{13}}{7} = \frac{72-5\sqrt{13}}{103}$$
$$\cos\psi = \frac{5}{13}\times\frac{6}{7} + \frac{12}{13}\times\frac{\sqrt{13}}{7} = \frac{30+12\sqrt{13}}{103}.$$

Lastly since $\sin\psi > 0$ and $\cos\psi > 0$ we have that ψ is in QI.

It also turns out that tangent has similar formulas, and fortunately, questions involving it are solved exactly like the previous two. The formulas are

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}, \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$$

There is amazing versatility in these formulas, especially the sum formulas for cosine, sine, and tangent. It turns out that by setting $\alpha = \beta$ we get what are called the double angle formulas.

Theorem 6 *Given an angle α, the following hold*

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$
3. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Problem Double Angles: Given α in QIV and $\cos\alpha = \dfrac{4}{13}$, find $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$

(Solution) Since we are given $\cos\alpha$, all we need to find is $\sin\alpha$ which can easily be done using the Pythagorean Identities,

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} \qquad \Rightarrow \qquad \sin\alpha = \sqrt{1-\left(\frac{-4}{13}\right)^2} = \frac{\sqrt{153}}{13}$$

The only thing you need to note is that we took the positive sign since we are in QIV. Now we just apply our formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\times\frac{\sqrt{153}}{13}\times\frac{4}{13} = \frac{8\sqrt{153}}{169} > 0$$
$$\cos 2\alpha = 2\cos^s\alpha - 1 = 2\left(\frac{4}{13}\right)^2 - 1 = \frac{-137}{169} < 0.$$

So two things, since sine is positive and cosine is negative, we have that 2α is in QII, and tangent can be found in the normal sense of taking the ratio, so

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{8\sqrt{153}}{-137}.$$

Following a similar method to the double angles, we can use the double angles and the sum formulas to find triple angles by using the substitution $3\alpha = 2\alpha + \alpha$.

Theorem 7 *Given an angle α the following hold:*

1. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
2. $\cos 3\alpha = 4\cos^3 - 3\cos\alpha$
3. $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

These formulas have similar applications to the previous ones are, but there are some other applications that the triple formulas allow us to solve. Namely, we can find exact values to new angle measures.

Problem Exact Values: Find the exact value of $\sin\frac{\pi}{5}$ and $\cos\frac{\pi}{5}$.

(Solution) Solving this problem relies on some rather neat "tricks," first of which is that

$$3\alpha + 2\alpha = \pi \qquad \Rightarrow \qquad 3\alpha = \pi - 2\alpha.$$

Meaning that cosine of either side is equivalent, and using the identity $\cos(\pi - a) = -\cos a$ we reach

$$\cos 3\alpha = \cos(\pi - 2\alpha) = -\cos 2\alpha \qquad \Rightarrow \qquad \cos 3\alpha + \cos 2\alpha = 0.$$

Now that we have this sum, we can use our double angle and triple angle identities for cosine to rewrite it in terms of $\cos\alpha$

$$4\cos^3 - 3\cos\alpha + 2\cos^2\alpha - 1 = 0 \qquad \text{let } x = \cos\alpha \text{ then} \qquad 4x^3 + 2x^2 - 3x - 1 = 0.$$

It turns out that $x = 1$ is a zero, plug it in to see, which means that $x - 1$ is a factor and by polynomial long division we can "remove" the factor to get a quadratic

$$\frac{4x^3 + 2x^2 - 3x - 1}{x - 1} = 4x^2 - 2x - 1.$$

Now we want to solve $4x^2 - 2x - 1 = 0$ which is accomplished using quadratic formula.

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{1 \pm \sqrt{5}}{4} \qquad \frac{1+\sqrt{5}}{4} > 0 \qquad \frac{1-\sqrt{5}}{4} < 0.$$

Now we have two solutions to our quadratic formula which means we have the following

$$\cos\alpha = \frac{1+\sqrt{5}}{4} \qquad \cos\alpha = \frac{1-\sqrt{5}}{4}.$$

Only one of these refers to the α we want, and we can easily tell that since $\pi/5$ is a quadrant one angle, that we want the positive solution. Hence

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} \qquad \text{other angle is } \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}.$$

Recall the identities for the double angles,

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}.$$

If we use the substitution of $2\alpha = \theta$, then we get what are called the half angle identities.

Theorem 8 *Given θ, the following hold:*

1. $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$
2. $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
3. $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$

Like the other applications, solutions to problems using the half-angle formulas are very similar. Consider the following:

Problem Half-Angle: Find $\sin\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ based on $\pi < \theta < \frac{3\pi}{2}$ and $\sin\theta = -3/5$.

(Solution) Since we are given sine, we need to find cosine. This is done using the fact that $\sin\theta$ gives us

$y = -3$ and $r = 5$, and by Pythagorean Theorem (and that the angle is QIII), we get $x = -\sqrt{5^2 - (-3)^2} = -4$. Hence $\cos\theta = -4/5$. Now we just apply our formulas

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+(-4/5)}{2}} = -\frac{1}{\sqrt{10}} \qquad \sin\frac{\theta}{2} = \sqrt{\frac{1-(-4/5)}{2}} = \frac{3}{\sqrt{10}}.$$

Now we know the signs on these based on the fact that $\pi < \theta < \frac{3\pi}{2}$ implies that $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$ which is in the second quadrant.

From earlier on we had four formulas for sums and differences of angles,

1. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
4. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

The fascinating part is that if you take the sums and differences of (1) & (2) and (3) & (4), we get fours formulas that takes sums of angles to products.

Theorem 9 *Given angles α and β, the following hold:*

1. $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
2. $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
3. $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$
4. $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

Granted this is one formulation of these identities, however there is a better way to view these identities that is more beneficial to how we will use them for this course. This is achieved by let $\alpha+\beta = u$ and $\alpha-\beta = v$, after using system of equations to solve for α and β we get that

$$\alpha = \frac{u+v}{2} \qquad \beta = \frac{u-v}{2}.$$

Theorem 10 *Given angles u and v, the following hold:*

1. $\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$
2. $\cos u - \cos v = 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{v-u}{2}\right)$
3. $\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$
4. $\sin u - \sin v = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$

These formulas are useful in solving equations similar to the following example.

Problem Sum to Product: Find all solutions to $\cos 4x + \sin 8x = 0$.

(Solution) Interesting enough, this is just an application of the previous theorem and utilization of the correct arc-function. So by (3) we get

$$\cos 4x + \sin 8x = 2\sin(6x)\cos(-2x) = 0.$$

Since we have two things that multiply to zero, then one of them must be zero. First we can have $\sin 6x = 0$

$$\sin(6x) = 0 \qquad \Rightarrow \qquad 6x = (-1)^k \arcsin(0) + \pi k \qquad \Rightarrow \qquad x = \frac{\pi k}{6}.$$

The other option we have is that $\cos(-2x) = 0$ which yields solutions

$$\cos(-2x) = 0 \qquad \Rightarrow \qquad -2x = \pm\arccos(0) + 2\pi k \qquad \Rightarrow \qquad x = \mp\frac{\pi}{4} - \pi k.$$

5 Chapter 5

This is the chapter, where things get, shall I say it, complex. This is the introduction to the complex plane and complex numbers. A **complex number** is a number of the form

$$z = a + bi \qquad a, b \text{ real numbers}$$

1. the symbol i is called the imaginary unit, $i = \sqrt{-1}$
2. the number a is called the real part of z, denoted $Re(z)$
3. the number b is called the imaginary part of z, denoted $Im(z)$
4. Note, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

Despite introducing the imaginary part, arithmetic for complex numbers is fairly straight forward. For example, assume we have complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then addition, subtraction, multiplication are as follows:

1. $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
2. $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$
3. $z_1 \times z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$ this comes from FOIL

Lets look at a few examples now.

Problem Complex Arithmetic: Given $z_1 = 1 - 3i$ and $z_2 = -2 + i$, find $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$.

(Solution)

$$z_1 + z_2 = (1 - 3i) + (-2 + i) = (1 - 2) + (-3 + 1)i = -1 - 2i$$
$$z_1 - z_2 = (1 - 3i) - (-2 + i) = (1 - (-2)) + (-3 - 1)i = 3 - 4i$$
$$z_1 z_2 = (1 - 3i)(-2 + i) = (1)(-2) + (1)(i) + (-3i)(-2) + (-3i)(i) = (-2 + 3) + (1 + 6)i = -1 + 7i$$

Before we are able to talk about division of complex numbers, we want to define two more quantities that are important to complex numbers. The first is the complex conjugate of a number, later on we will see this as a type of reflection.

Complex Conjugate of a complex number $z = a + bi$ is, $\bar{z} = a - bi$. Which just changes the sign on the imaginary part of our complex number. It also follows that $\bar{\bar{z}} = z$.

The other number is something we have seen plenty of times in the past, but in other disguises. This is called the modulus of z.

Modulus of $z = a + bi$ is the non-negative real number $|z| = \sqrt{a^2 + b^2}$. Sometimes you will see this called the absolute value of z. If you were wondering, this looks similar to the length of the vector $[a, b]$, and you will see why later.

There is actually a nice tie between the modulus and the conjugate in that for any complex number $z = a + bi$ we have

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

Now that we have these new tools we are able to talk about division of two complex numbers. Assuming the denominator is non-zero, it follows that

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} \qquad \Leftrightarrow \qquad \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{-a_1 b_2 + a_2 b_i}{a_2^2 + b_2^2} i$$

Problem Complex Division: Divide the numbers $z_1 = 2 - 4i$ by $z_2 = 1 + 3i$.

(Solution) According to the formula we get

$$\frac{2 - 4i}{1 + 3i} = \frac{(2)(1) + (-4)(3)}{1^2 + 3^2} + \frac{(2)(3) + (-4)(1)}{1^2 + 3^2} = -1 + \frac{1}{5} i$$

In past classes, we had times when we would have a polynomial, say $x^2 + 1$, and we would be attempt to factor it but were not able to. This changes now that we are working over the complex numbers, $\mathbb{C}$, with the following theorem

Theorem 11 *Any polynomial with complex coefficients has at least one complex zero. Meaning the equation*

$$P(x) = 0$$

for some polynomial $P(x)$ with complex coefficients has a zero of the form $x = a+bi$. Moreover, any polynomial of with degree n completely factors as a product of n factors:

$$P(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_n)$$

where a_n is the leading coefficients and $z_1, \ldots, z_n$ are zeroes.

The power of this theorem is that the polynomial from earlier x^2+1 has factors over the complex plane, meaning it can be written as

$$x^2 + 1 = (x - i)(x + i).$$

This now gives us more problems to practice.

Problem Factoring: Find the solutions to $P(x) = 3x^2 + 3x + 5$.

(Solution) By our previous theorem this quadratic will factor as $P(x) = 3(x - z_1)(x - z_2)$ where z_1 and z_2 are solutions to $P(x) = 0$. So we need to solve the equation $3x^2 + 3x + 5 = 0$ which is done using the quadratic formula

$$x = \frac{-3 \pm \sqrt{3^2 - 4(3)(5)}}{2(3)} = \frac{-3 \pm \sqrt{-51}}{6} = \frac{-3 \pm i\sqrt{51}}{6}.$$

Now that we have the two solutions it follows that

$$P(x) = 3\left(x - \frac{-3 + i\sqrt{51}}{6}\right)\left(x - \frac{-3 - i\sqrt{51}}{6}\right)$$

When the coefficients are all real, the solution is just like we encounter in College Algebra, but we can now deal with the square roots of negative numbers. Sometimes we encounter situations where one of the coefficients is complex, which changes the method slightly. We will showcase this using an example.

Problem Complicated Factoring: Lets solve the quadratic $x^2 - 3x + 2 + 2i = 0$.

(Solution) Recall from College Algebra that the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$. This is the first thing we need to solve our situation.

$$D = (-3)^2 - 4(1)(2 + 2i) = 9 - 8 - 8i = 1 + 8i.$$

Since the discriminant is complex, we need to find a solution to the equation

$$d^2 = 1 + 8i.$$

Since d has to be complex, we can assume it looks like $d = u + vi$. Thus it follows that

$$d^2 = (u + vi)^2 = u^2 - v^2 + 2uvi = 1 + 8i \quad \Rightarrow \quad u^2 - v^2 = 1 \quad \text{and} \quad 2uv = 8.$$

It then follows that by solving for v, $v = \frac{4}{u}$ and substitution into the equation for the real part that

$$u^2 - \frac{16}{u^2} = 1 \quad \Rightarrow \quad u^4 - u^2 - 16 = 0$$

which is quadratic under the substitution $t = u^2$. Hence

$$t^2 - t - 16 = 0 \quad t = \frac{1 \pm \sqrt{1 - 4(1)(-16)}}{2} = \frac{1 \pm \sqrt{65}}{2}.$$

This gives us two solutions to $t = u^2$, but since everything is real for this substitution, then $\frac{1-\sqrt{65}}{2}$ is not valid since it is negative. Thus

$$u^2 = \frac{1-\sqrt{65}}{2} \quad \Rightarrow \quad u = \sqrt{\frac{1-\sqrt{65}}{2}} \quad \text{and back sub on v yields } v = \frac{4}{u} = \frac{4}{\sqrt{\frac{1-\sqrt{65}}{2}}}.$$

For the sake of convience, we will use u and v instead of their numerical forms for easy. Now that we know $d = u + vi$, then it follows that

$$x = \frac{3 \pm (u+vi)}{2} \quad \text{yielding the two solutions: } x_1 = \frac{3+u+vi}{2} \quad x_2 = \frac{3-u-vi}{2}$$

where u and v are stated earlier.

This is a complicated style of problem, but the steps we took allow us to now solve any equation of the form $z^2 = \omega$ for any complex ω. Which is similar to how we found d in the previous problem. The overview of the problem, solve $z^2 = a + bi$ is as follows

1. Assume $z = u + vi$ is a solution,
2. Then for it to be a solution we must have $u^2 - v^2 = a$ and $2uv = b$.
3. Substitute $v = \frac{b}{2u}$ and solve $u^4 - au^2 - \frac{b^2}{4} = 0$
4. Using the substitution $t - u^2$, solve $t^2 - at - \dfrac{b^2}{4} = 0$, using quadratic formula.
5. Chose a positive solution t, then $u = \sqrt{t}$ and $v = \frac{b}{2\sqrt{t}}$.

Problem Application: Solve $z^2 = 3 - 4i$.

(Solution) Using our steps, assume that $z = u + vi$ is a solution then by step (2) we have that

$$u^2 - v^2 = 3 \quad \text{and} \quad 2uv = -4 \quad \Rightarrow \quad uv = -2.$$

Now using the substitution that $v = \frac{-2}{u}$ we get the following

$$u^2 - v^2 = 3 \quad \Rightarrow \quad u^2 - \left(\frac{-2}{u}\right) = 3 \quad \Rightarrow \quad u^4 - 3u^2 - 4 = 0.$$

By letting $t = u^2$ we get

$$t^2 - 3t - 4 = 0 \quad \Rightarrow \quad t = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2} = \frac{3 \pm 5}{2}.$$

We see that the negative solution does not work so

$$u = \sqrt{4} = 2 \quad v = -1.$$

Thus $z = 2 - i$ is the solution we want.

An interesting feature of i is that it has its own "periodicity" which we can observe as follows

$$i = \sqrt{-1} \quad \rightarrow \quad i^2 = -1 \quad \rightarrow \quad i^3 = i^2 i = -i \quad \rightarrow \quad i^4 = i^2 i^2 = 1.$$

It then turns out that i^a for any number a can be found by division, meaning $a = 4b + r$ for some integer b and $0 \le r \le 3$, which tells us $i^a = (i^4)^b i^r = i^r$. In other words, the solution is just i^r where r is the remainder from dividing a by 4.

Problem Powers of i: Find i^{317}

(Solution) Dividing 317 by 4 yields $317 = 4(79) + 1$ so the remainder is one. Thus

$$i^{317} = (i^4)^{79} i^1 = 1^{79} i = i.$$

Polar Plane:
The interesting thing about a complex number $z = a + bi$ is that it can be represented as the ordered pair $P(a, b)$ and the modulus, $|z|$ is equivalent to the distance from the origin to the point P. Using this construction we see that the conjugate of z, $\bar{z} = a - bi$, is just a reflection of the point over the x-axis since $\bar{P}(a, -b)$.
Since we can represent complex numbers as points on the plane (and also vectors), we can use what we learned in a previous sections to find polar coordinate representations of these complex numbers. Recall that polar form is just (r, θ) so for a complex number z we have:

Polar Representation of a complex number z is an presentation of it as a product

$$z = r(\cos\theta + i\sin\theta) = r\text{cis }\theta$$

where r is uniquely determined by

$$r = |z| = \sqrt{(Rez)^2 + (Imz)^2}$$

$$\theta = (\text{sign Im z})\arccos\left(\frac{Rez}{|z|}\right)$$

provided $z \neq 0$, this is the exact formula from chapter three, just changed to fit the Complex number language. Similar to previous, if $Imz = 0$ assume sign is $(+)$. Since we are dealing with angles, you should remember that we can have multiple representations of the same number by adding on integer multiples of 2π. In essence the θ we find using the formula above is called the principle angle.

Problem Conversion: Given $z = 2 - 2i$ find the polar representations and given $z = 2\text{cis }\dfrac{\pi}{4}$, find the rectangular representation.

(Solution) For z we need to find $|z|$ and θ which is done as follows

$$|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}, \qquad \theta = -\arccos\frac{2}{2\sqrt{2}} = \frac{-\pi}{4}.$$

So $z = 2\sqrt{2}\text{cis }\left(\dfrac{-\pi}{4}\right)$. Which is the principle representation, all representations is of the form

$$z = 2\sqrt{2}\text{cis }\left(\frac{-\pi}{4} + 2\pi k\right).$$

For the other part, it goes as follows

$$z = 2\text{cis }\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\sin\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}.$$

There are some fun facts for complex numbers of the form cis θ, ie complex numbers on the unit circle.

$$|\text{cis }\theta| = 1, \qquad \bar{\text{cis}}\ \theta = \text{cis }(-\theta) = \frac{1}{\text{cis }\theta}.$$

The nice thing about polar representation of complex numbers is that the multiplication and division are a lot nicer, given $z_1 = |z_1|\text{cis }\theta_1$ and $z_2 = |z_2|\text{cis }\theta_2$, then the operations are as follows:

$$z_1 z_2 = |z_1||z_2|\text{cis }(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\text{cis }(\theta_1 - \theta_2).$$

Problem Polar Multiplication & Division: Given $z_1 = 2\text{cis }\frac{\pi}{4}$ and $z_2 = 3\text{cis }\frac{-\pi}{8}$, find $z_1 z_2$ and $\frac{z_1}{z_2}$.

(Solution)

$$z_1 z_2 = (2\times 3)\text{cis}\left(\frac{\pi}{4} + \frac{-\pi}{8}\right) = 6\text{cis}\,\frac{\pi}{8} \quad \text{and} \quad \frac{z_1}{z_2} = \frac{2}{3}\text{cis}\left(\frac{\pi}{4} - \frac{-\pi}{8}\right) = \frac{2}{3}cis\frac{3\pi}{8}.$$

It turns out that repeated multiplication of the same complex number yields a formula, called De Moivre's formula

Theorem 12 *For any complex number z and N positive integer*

$$(r\cos\theta + i\sin\theta)^N = r^N(\cos N\theta + i\sin N\theta).$$

This can be adapted to negative integers by

$$(r\cos\theta + i\sin\theta)^{-N} = r^{-N}(\cos N\theta - i\sin N\theta).$$

This theorem allows us to take complex numbers to large powers easily provided we are in our polar representation.

Problem DeMoivre's: Find $(1+i)^{20}$.

(Solution) First thing we want to do is convert $z = 1+i$ to its polar form which is done as follows:

$$|z| = \sqrt{1^2+1^2} = \sqrt{2} \quad \text{and} \quad \theta = +\arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

Thus $1+i = \sqrt{2}\text{cis}\,\frac{\pi}{4}$, and now that we have the correct formula, we apply De Moivre's.

$$(1+i)^{20} = \left(\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) = 1024(\cos 5\pi + i\sin 5\pi) = 1024(-1+0i) = -1024.$$

The true power of De Moivre's formula is in solving equations of the form $z^N = \omega$ for any complex number ω. However, we are going to focus on a special sub group of these polynomials by letting $\omega = 1$. The solutions to $z^N = 1$ are called the N^{th} roots of unity.

Theorem 13 *For any integer $N \geq 2$, the N distinct roots of unity are given by*

$$u_k = \cos\frac{2k\pi}{N} + i\sin\frac{2k\pi}{N}, \qquad k = 0, 1, \ldots, N-1.$$

Moreover, these are the only solutions to $z^N - 1 = 0$ and it follows that

$$z^N - 1 = (z-u_0)(z-u_1)\cdots(z-u_{N-1}).$$

Problem Roots of Unity: Find the seventh roots of unity.

(Solution) Since we want to find the seventh roots of unity, we want to solve $z^7 - 1 = 0$ which is a direct application of the formula above with $N = 7$.Thus by the first part we have

$$u_k = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7} \qquad k = 0, 1, 2, 3, 4, 5, 6.$$

- $u_0 = \cos\frac{2(0)\pi}{7} + i\sin\frac{2(0)\pi}{7} = 1$
- $u_1 = \cos\frac{2(1)\pi}{7} + i\sin\frac{2(1)\pi}{7} = \text{cis}\,\frac{2\pi}{7}$
- $u_2 = \cos\frac{2(2)\pi}{7} + i\sin\frac{2(2)\pi}{7} = \text{cis}\,\frac{4\pi}{7}$
- $u_3 = \cos\frac{2(3)\pi}{7} + i\sin\frac{2(3)\pi}{7} = \text{cis}\,\frac{6\pi}{7}$

- $u_4 = \cos\frac{2(4)\pi}{7} + i\sin\frac{2(4)\pi}{7} = \text{cis }\frac{8\pi}{7}$
- $u_5 = \cos\frac{2(5)\pi}{7} + i\sin\frac{2(5)\pi}{7} = \text{cis }\frac{10\pi}{7}$
- $u_6 = \cos\frac{2(6)\pi}{7} + i\sin\frac{2(6)\pi}{7} = \text{cis }\frac{12\pi}{7}$

Theorem 14 *If $N \geq 2$ is an integer and ω is a non-zero complex number, then the N distinct roots of ω are given by*

$$z_k = \sqrt[n]{|\omega|}\left(\cos\frac{\theta + 2k\pi}{N} + i\sin\frac{\theta + 2k\pi}{N}\right), \qquad k = 0, 1, \ldots, N-1.$$

Where θ is the argument from finding the polar representation of ω.

Problem General Roots: Find the cubic roots of $1 + i$.

(Solution) First we need to convert $1 + i$ to polar form, which was done previously, so $1 + i = (\sqrt{2}, \pi/4)$. Thus $N = 3$ by the General Roots theorem we have

$$z_k = \sqrt[3]{\sqrt{2}}\left(\cos\frac{\frac{\pi}{4} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{4} + 2k\pi}{3}\right) = \sqrt[4]{2}\left(\cos\frac{(8k+1)\pi}{12} + i\sin\frac{(8k+1)\pi}{12}\right), \qquad k = 0, 1, 2.$$

- $z_0 = \sqrt[4]{2}\left(\cos\frac{(8(0)+1)\pi}{12} + i\sin\frac{(8(0)+1)\pi}{12}\right) = \sqrt[4]{2}\text{cis }\frac{\pi}{12}$
- $z_1 = \sqrt[4]{2}\left(\cos\frac{(8(1)+1)\pi}{12} + i\sin\frac{(8(1)+1)\pi}{12}\right) = \sqrt[4]{2}\text{cis }\frac{3\pi}{4}$
- $z_2 = \sqrt[4]{2}\left(\cos\frac{(8(2)+1)\pi}{12} + i\sin\frac{(8(2)+1)\pi}{12}\right) = \sqrt[4]{2}\text{cis }\frac{17\pi}{12}$

Application: We are going to apply the theorem we just learned to find a nice way to calculate square roots, i.e. solutions to $z^2 - \omega = 0$ for complex ω. The two roots of ω if it is NON-REAL are $\pm z$ where

$$z = \sqrt{\frac{|\omega| + \text{Re }\omega}{2}} + (\text{sign of Im }\omega)i\sqrt{\frac{|\omega| - \text{Re }\omega}{2}}$$

However if ω is real then the square roots are:

1. $\pm\sqrt{\omega}$ if $\omega \geq 0$
2. $\pm i\sqrt{|\omega|}$ if $\omega \leq 0$

Problem Square Root Formula: Find the two square roots of $\omega = 9 - 12i$.

(Solution) It follows that Re $\omega = 9$ and Im $\omega = -12$, the only thing we need besides this information is the modulus which is

$$|\omega| = \sqrt{9^2 + (-12)^2} = \sqrt{81 + 144} = \sqrt{225} = 15.$$

Further we see the Im ω is negative, so the roots are as follow

$$z = \sqrt{\frac{|\omega| + \text{Re }\omega}{2}} + (\text{sign of Im }\omega)i\sqrt{\frac{|\omega| - \text{Re }\omega}{2}} = \sqrt{\frac{15+9}{2}} + (-)i\sqrt{\frac{15-9}{2}} = \sqrt{12} - i\sqrt{3}.$$

Thus both solutions are $z_+ = \sqrt{12} - i\sqrt{3}$ and $z_- = -(\sqrt{12} - i\sqrt{3}) = -\sqrt{12} + i\sqrt{3}$

6 Chapter 6

In this chapter we are looking at quadratic equations in two variables of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

for constants a, b, c, d, e, f. What is happening here is that we have two cones stacked on top of each other with the tips touching, and we take slices in varying directions we can create different shapes, namely, a circle, ellipse, parabola, and a hyperbola.

Parabolas: When we talk about parabolas, you are probably thinking equations of the form $y = ax^2 + bx + c$, and that is similar to what we are dealing with, however we aren't in 2D plane anymore but rather a 3D space.

The way these shapes are described is using what is called a **geometric locus** which is essentially a set of points that shares a common geometric property. For example the geometric locus of a circle centered at (h, k) of radius r is that all the points are distance r from the point (h, k).

Formal Definition of Parabola is given a line $\mathcal{D}$ and some point F not on $\mathcal{D}$, the parabola with focus F and directrix $\mathcal{D}$ is the set of points P on the plane that are equal distance from F and $\mathcal{D}$. Meaning $d(P, F) = d(P, \mathcal{D})$.

Besides the focus, there is also the vertex, V, of a parabola which is the closest point to the directrix, and the **focal axis** is the line through F and V.

Standard Position: A parabola $\mathcal{P}$ is in standard position with vertex $V(x_v, y_v)$ and focus $F(x_f, y_f)$ if

1) We have a horizontal directrix, i.e. $\mathcal{D} := y_D = d$ for some number d

- Focus coordinates are $x_f = x_v$ and $y_f = y_v + p$
- Significant coordinate of directrix is $y_D = y_v - p$
- Focal axis is vertical, so $p = y_v - y_D = y_f - y_v = \frac{1}{2}(y_f - y_d)$
- Has standard form $(x - x_v)^2 = 4p(y - y_v)$

2) If the parabola has a vertical directrix, i.e $\mathcal{D} := x_D = d$ for some number d.

- Focus coordinates are $y_f = y_v$ and $x_f = x_v + p$
- Significant coordinate of directrix is $x_D = x_v - p$
- Focal axis is horizontal $p = x_v - x_D = x_f - x_v = \frac{1}{2}(x_f - x_d)$
- Has standard form $(y - y_v)^2 = 4p(x - x_v)$

The variable p in the above definitions is called the **focal parameter** and is given by

$$|p| = d(V, F) = d(V, \mathcal{D}) = \frac{1}{2}d(F, \mathcal{D})$$

NOTE: Horizontal directrix $\to$ change in y coordinate of focus, while vertical directrix $\to$ change in x coordinate of focus.

Problem Geometric to Equation: Find the parabola with directrix $x = 2$ and vertex $V(3, 1)$.

(Solution) Since the directrix is $x = 2$ we have that it is a vertical directrix and therefore a horizontal focal axis. Thus $x_v = 3$ and $y_v = 1$, so we can substitute what we know into the equation to yield

$$(y - 1)^2 = 4p(x - 3).$$

Since this is a horizontal focal axis we have that $p = x_v - x_D = 3 - 2 = 1$. Thus the standard position equation is

$$(y - 1)^2 = 4(x - 3).$$

Problem Geometric to Equation: Find the parabola with directrix $y = 2$ and focus $F(1, -1)$.

(Solution) Since the directrix is $y = 2$ we have that it is a horizontal directrix and therefore a vertical focal axis. Thus the vertex are $x_v = 1$, vertical focal axis so x-coord does not change, and $y_v = -1 + 2\frac{1}{2} = 0$, since a vertical focal axis means the y-coord does change. Thus $V(1, 0)$, and by our earlier formula for p we have that $p = y_v - y_D = 0 - 2 = -2$. Therefore the equation is given as

$$(x - 1)^2 = 4(-2)(y - 0) \qquad \Rightarrow \qquad (x - 1)^2 = -8(y - 0)$$

We have looked at going from geometric data to an equation, now we want to work the opposite direction. Meaning if given an equation, can we find the focus, vertex, focal parameter, etc.

Equations:

- An equation of the form $y = ax^2 + bx + c$ has a vertical focal axis, and therefore has a horizontal directrix.

  Has form $y = a(x - h)^2 + k$ where $V(h, k)$.
- An equation of the form $x = ay^2 + by + c$ has a horizontal focal axis, and therefore has a vertical directrix.

  Has form $x = a(y - k)^2 + h$ where $V(h, k)$.
- For either form however we have $a = \frac{1}{4p}$ and $p = \frac{1}{4a}$

Problem Equation to Geometric Consider the equation $x = 4y^2 + 24y - 8$.

(Solution) The first thing to note is that we need to use completing the square to get it into the correct form. Moreover, since it is $x = \#$ we have that it has a horizontal focal axis or vertical directrix. So we need to complete the square for y,

$$x = 4(y^2 + 6y) - 8 \qquad \Rightarrow x = 4(y^2 + 6y + 9 - 9) - 8 \qquad \Rightarrow x = 4(y + 3)^2 - 9(4) - 8.$$

Thus the equation is $x = 4(y + 3)^2 - 44$, which tells us the vertex is $V(-44, -3)$. Once we find p, we can find the remaining information

$$p = \frac{1}{4a} = \frac{1}{4(4)} = \frac{1}{16}.$$

Therefore

$$x_F = x_v + p = -44 + \frac{1}{16} = \frac{-703}{16} \qquad x_D = x_v - p = -44 - \frac{1}{16} = \frac{-705}{16}.$$

Therefore the focus is $F(\frac{-703}{16}, -3)$ and $x_D = \frac{-705}{16}$.

Ellipses has a geometric locus defined by, given two distinct points F_1 and F_2 and some real number $D > d(F_1, F_2)$, the ellipse with foci F_1, F_2 and width D is the set of all points P that satisfy

$$d(P, F_1) + d(P, F_2) = D.$$

There is special language that accompanies the ellipse as follows

1. The line that passes through the foci is the **major axis**, and the two points that the major axis intersects the ellipse are called the **major points**, V_1 and V_2.
2. the line perpendicular to the major axis is the **minor axis** and the two points the minor axis intersects the ellipse at are called the **minor points**, M_1 and M_2.
3. The **center** is the point where the major and minor axis intersect.
4. **Major Radius** denoted a, $a = d(Z, V_1) = d(Z, V_2) = \frac{1}{2}d(V_1, V_2)$
5. **Minor Radius** denoted b, $b = d(Z, M_1) = d(Z, M_2) = \frac{1}{2}d(M_1, M_2)$
6. **Focal distance** denoted c, $c = d(Z, F_1) = d(Z, F_2) = \frac{1}{2}d(F_1, F_2)$

7. The last three variables satisfy $c^2 = a^2 - b^2$

8. The **eccentricity** of an ellipse is the ration $e = \frac{c}{a}$

Just like with parabolas, we are interested in ellipses in standard position. So for a ellipse we consider the following definitions.

Standard Position: An ellipse $\mathcal{E}$ is in standard position, with center $Z(x_z, y_z)$, major radius a, minor radius b and focal distance c, then

1) We have a horizontal major axis/ vertical minor axis, then the standard form is $\dfrac{(x-x_z)^2}{a^2} + \dfrac{(y-y_z)^2}{b^2} = 1$

- The vertices are $(x_z \pm a, y_z)$
- The foci are $(x_z \pm c, y_z)$
- Minor points are $(x_z, y_z \pm b)$

2) We have a vertical major axis/ horizontal minor axis, then the standard form is $\dfrac{(x-x_z)^2}{b^2} + \dfrac{(y-y_z)^2}{a^2} = 1$

- The vertices are $(x_z, y_z \pm a)$
- The foci are $(x_z, y_z \pm c)$
- Minor points are $(x_z \pm b, y_z)$

Like with parabolas, we want to be able to find the equation of the ellipse using geometric data, and hopefully vice-versa. The only other formulas that are useful for this task are

$$\frac{b}{a} = \sqrt{1-e^2} \qquad \frac{c}{b} = \frac{e}{\sqrt{1-e^2}}$$

Problem Geometric to Equation: Find the equation of the ellipse with vertices $V_1(-3,1)$, $V_2(-3,6)$ and a focus at $F(-2,2)$.

(Solution) By looking at the x-coordinates of the vertices, we see they are both set at $x = -3$, so we have a vertical major axis and a horizontal minor axis. Hence our equation looks something like

$$\frac{(x-x_z)^2}{b^2} + \frac{(y-y_z)^2}{a^2} = 1.$$

From this, since the center is the midpoint between the vertices, we have the x-coordinate is still -3, ie $x_z = -3$ and the y-coordinate is found using the midpoint formula.

$$y_z = \frac{1}{2}(6+1) = \frac{7}{2} \qquad \Rightarrow \qquad Z\left(-3, \frac{7}{2}\right).$$

Lastly we need to find the variables a, b, c which can be done using distance formulas

$$a = d(Z, V_1) = d(Z, V_2) = 6 - \frac{7}{2} = \frac{5}{2} \qquad c = d(Z, F_1) = d(Z, F_2) = \frac{7}{2} = 2 = \frac{3}{2}.$$

It then follows that $b^2 = a^2 - c^2 = (5/2)^2 - (3/2)^2 = 4$.Therefore the equation we want is

$$\frac{(x+3)^2}{4} + \frac{\left(y - \frac{7}{2}\right)^2}{\frac{25}{4}} = 1 \qquad \text{or} \qquad \frac{(x+3)^2}{4} + \frac{4\left(y - \frac{7}{2}\right)^2}{25} = 1.$$

For the reverse direction, we are going to tackle the problem of getting geometric data from equations by using intuition and understanding of the forward direction.

Problem Equation to Geometric: Consider the equation $3x^2 + 15x + 4y^2 - 64y - 50 = 0$.

(Solution) The first thing you want to do is get it into one of the forms from earlier and this is done by completing the square on both the x and y variable.

$$\begin{aligned}
3x^2 + 15x + 4y^2 - 64y &= 50 \\
3\left(x^2 + 5x + \frac{25}{2} - \frac{25}{2}\right) + 4(y^2 - 16y + 64 - 64) &= 50 \\
3\left(x + \frac{5}{2}\right)^2 - 3 \cdot \frac{25}{2} + 4(y-8)^2 - 64 \cdot 4 &= 50 \\
3\left(x + \frac{5}{2}\right)^2 + 4(y-8)^2 &= 50 + 64 \cdot 4 + 3 \cdot \frac{25}{2} \\
3\left(x + \frac{5}{2}\right)^2 + 4(y-8)^2 &= \frac{687}{2} \\
\frac{\left(x + \frac{5}{2}\right)^2}{4 \cdot \frac{687}{2}} + \frac{(y-8)^2}{\frac{3 \cdot 687}{2}} &= 1 \\
\frac{\left(x + \frac{5}{2}\right)^2}{1374} + \frac{(y-8)^2}{1030.5} &= 1
\end{aligned}$$

From this calculation we get the center $Z\left(\frac{-5}{2}, 8\right)$ and we also get the variables a and b. Since $1374 > 1030.5$, then $a = \sqrt{1374}$ and $b = \sqrt{1030.5}$, yes this follows simply from one number being larger. This tells us we have a horizontal major axis. Note, if they are equal, we are dealing with a circle. Now that we have a and b we can find c and e.

$$c = \sqrt{a^2 - b^2} = \sqrt{1374 - 1030.5} - \sqrt{343.5}, \qquad e = \frac{c}{a} = \frac{\sqrt{343.5}}{\sqrt{1030.5}}.$$

The last pieces of information we can obtain follows from how we found the foci, vertices, and minor points from earlier.

$$F(x_z \pm c, y_z) = F\left(\frac{-5}{2} \pm \sqrt{343.5}, 8\right) \qquad V(x_z \pm a, y_z) = V\left(\frac{-5}{2} \pm \sqrt{1030.5}, 8\right) \qquad M(x_z, y_z \pm b) = M\left(\frac{-5}{2}, 8 \pm \sqrt{1030.5}\right).$$