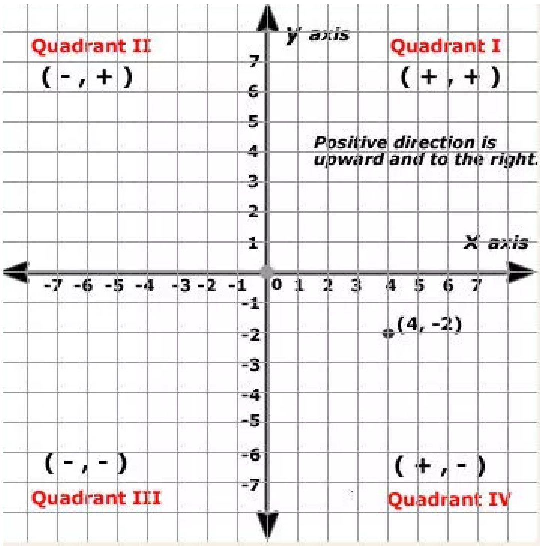
The purpose of these notes is to provide a concise overview of the material covered in MATH 1001 at Georgia State University. Note that the chapters are presented in order that they are presented in the course due to exam scheduling.

Chapter 7

7.1

For chapter 7 we are looking at functions, specifically linear functions in the first couple of sections and more complicated functions in section 7.6. However, to start off let us discuss the Cartesian coordinate system.

This Cartesian plane is able to show information using points, represented by **ordered pairs** (a, b), where a is the x-coordinate and b is the y-coordinate. The plane itself is split into four quadrants labelled as follows:



We are interested in the relationship between two quantities that can be expressed as an equation in two variables. Note that the variables can be anything, alphabet, Greek, etc, however it is fairly common to just use x and y. For example

$$y = 4x - 3x^2 \qquad 3x - 4y = 2$$

are both examples of equations in two variables. For equations, a **solution** is an ordered pair that when substituted into the equation it holds true.

Example 1. *Consider the equation $y = 4x - 3x^2$, check if the ordered pairs $(0, 0)$, $(1, 1)$, $(2, -4)$, and $(1, 2)$ are solutions.*

To check this we just substitute each number into their corresponding variable and check if the statement is true.

$(0,0)$	$0 = 4(0) - 3(0)^2$	*true*
$(1,1)$	$1 = 4(1) - 3(1)^2$	*true*
$(2,-4)$	$-4 = 4(2) - 3(2)^2$	*true*
$(1,2)$	$2 = 4(1) - 3(1)^2$	*false*

Therefore the solutions are $(0, 0)$, $(1, 1)$, and $(2, -4)$.

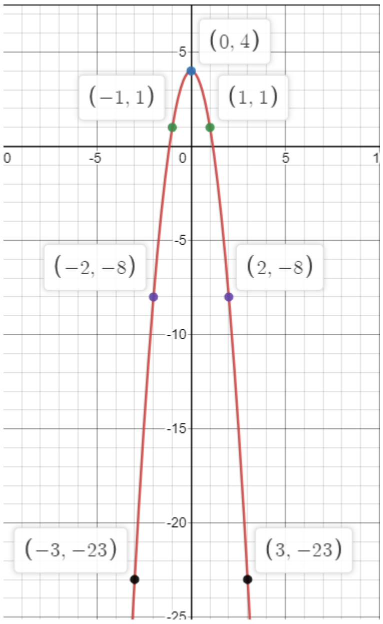
However, one should note that we can create as many solutions as possible by changing the values of the x-coordinate and solving for the corresponding y. Ordered pairs created in this manner are solutions to the given equation.

Definition 2. *The **graph** of an equation is the set of all points whose coordinates satisfy the equation.*

The easiest method to sketch the graph of an equation is called point-plotting method. This requires one to find a few points, i.e. solutions, and connect the dots.

Example 3. *Plot the graph* $y = 4 - 3x^2$*:*

x	*y*	*(x,y)*
-3	$4 - 3(-3)^2 = -23$	$(-3, -23)$
-2	$4 - 3(-2)^2 = -8$	$(-2, -8)$
-1	$4 - 3(-1)^2 = 1$	$(-1, 1)$
0	$4 - 3(0)^2 = 4$	$(0, 4)$
1	$4 - 3(1)^2 = 1$	$(1, 1)$
2	$4 - 3(2)^2 = -8$	$(2, -8)$
3	$4 - 3(3)^2 = -23$	$(3, -23)$



Example 4. *The toll road costs* $\$3.50$ *per use. Frequent drivers may purchase a pass for* $\$20$ *per month and in doing so, the toll drops to* $\$1.50$ *per use. Let* x *be the number of uses of the toll and* y *be the monthly cost, find equations for the cost of using the toll with and without a toll pass.*

Looking at the information given, the cost is determined by the number of times we use the toll times the the cost for each use. The only thing extra is whether or not we bought a toll pass. Therefore the cost without the pass is

$$y = 3.5x$$

and the cost with a pass is

$$y = 20 + 1.5x.$$

Consider the following chart:

x	$y = 3.5x$	$y = 20 + 1.5x$
0	0	20
2	7	23
4	14	26
6	21	29
8	28	32
10	35	35

Notice that at 10 uses the two equations are equal, i.e. the cost of using the toll with or without a pass is $\$35$*, this is where the the two equations intersect. It also tells us that if we use the toll less than 10 times, it is cheaper to not buy the toll pass, but if we use the toll more than 10 times then we should buy the pass.*

If an equation in two variables as the property that for each value of x there is exactly **one** y, we say that y is a **function** of x. Typically we see the notation $y = f(x)$ where f is the function's name, so recalling the last example we could say

$$y = 3.5x \qquad\qquad f(x) = 3.5x$$

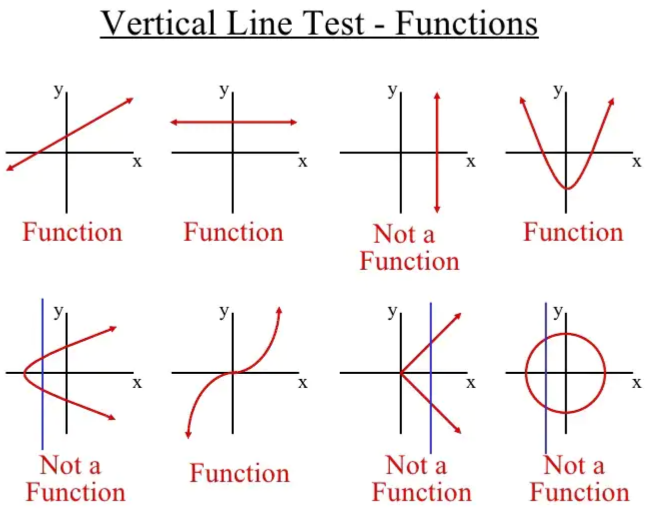
The benefit of this notation is that if I want to check what the value of the function is when $x = 10$, all we need to do is

$$f(10) = 3.5(10) = 35.$$

The only "change"that occurs when using function notation is that our ordered pairs go from (x, y) to $(x, f(x))$.

Most of the the equations we will deal with in this course will be functions, however not all graphs represent functions. We have a nice test to determine whether a graph is a function or not. The **vertical line test** determines this for us. It is stated as follows:

If any vertical line intersects a graph in more than one point, then the graph does not define a function y of x.



7.2

A function of the form

$$f(x) = ax + b$$

where a and b are any real numbers is called a **linear function**. These functions get this name since they are our straight lines.

The nice thing about linear functions is that any two points on the line determine the entire graph, and in this class there are two sets of points that we are interested in, the x-intercepts and the y-intercept.

1. x-intercept is the ordered pair where the graph crosses the x-axis and can be found by letting $y = 0$ and solving for x.
2. y-intercept is the ordered pair where the graph crosses the y-axis and can be found by letting $x = 0$ and solving for y.

For linear functions we are interested in **slope** of the line, which is defined as the steepness of the line. To rigorously defined it, consider the two points (x_1, y_1) and (x_2, y_2). The slope through the two points, commonly denoted m, is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}.$$

Example 5. *Find the slope through the points* $(-3, -1)$ *and* $(-2, 4)$.

Here we will let $(x_2, y_2) = (-3, -1)$ *and* $(x_1, y_1) = (-2, 4)$*, therefore the definition of the slope yields*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 4}{-3 - (-2)} = \frac{-5}{-1} = 5$$

In terms of slopes there are four types of slopes depending on whether $m > 0$, $m < 0$, $m = 0$, or m is undefined. (insert picture)
Lastly, there are three ways we can represent linear functions:

1. Slope-intercept form: $y = mx + b$, and this form gives us the slope m and the y-intercept $(0, b)$.

2. Point-slope form: $y - y_0 = m(x - x_0)$, and this form gives us the slope m and a point on the line (x_0, y_0).
3. General Form: $ax + by = c$, and this form gives the slope $m = -a/b$, y-intercept $(0, c/b)$ and x-intercept $(c/a, 0)$.

In terms of applications for linear functions, it is very common to refer to slope as a **rate of change**.

Example 6. *In the United States of America,* 65% *of women and* 45% *of men were married in 1970. Later on it was found out that* 21% *of women and* 11% *of men were married in 2010.*

Find the rate of change for women between the years 1970 and 2010.

For this question we need to find the slope between the points $(1970, 65\%)$ *and* $(2010, 21\%$ *which can be done using our formula for slope:*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{21 - 65}{2010 - 1970} = \frac{-44}{40} = -1.1\%/year.$$

Since the rate of change is negative, it is a decrease, and the units for the rate of change are the units of the y*-axis (%) divided by the units of the* x*-axis (years). Therefore the marriage rate for women between 1970 and 2010 is decreasing at a rate of* 1.1% *per year.*

7.3

For this section we are looking at solving systems of linear equations. When we have two equations of the form $ax + by = c$, this is called a **system of linear equations in two variables.** Typically represented as:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

A solution to the system is an ordered pair that satisfies both equations simultaneously.

Example 7. *Determine if* $(1, 2)$ *and* $(0, 2)$ *are solutions to the system.*

All we need to do is check if the points satisfy both equations, if it does not satisfy one, then it is not a solution to the system.

$$\begin{cases} 2x - 3y = -4 \\ 2x + y = \ 4 \end{cases}$$

Checking $(1, 2)$*:*

$$\begin{cases} 2(1) - 3(2) = 2 - 6 = -4 \\ 2(1) + (2) = \qquad 4 \end{cases}$$

So it is a solution to the system. Now for $(0, 2)$*:*

$$\begin{cases} 2(0) - 3(2) = 0 - 6 \neq -4 \\ 2(0) + (2) = \ 0 + 2 \neq 4 \end{cases}$$

Since neither equation is satisfied, $(0, 2)$ *is not a solution. Note, we only need the point to not satisfy one equation for it to note be a solution.*

One way to find the solutions to a system of linear equations is for you to graph the lines and find their intersection point. That point is the solution. However there are other methods to solving these problems.

Solving by Substitution:

1. Pick an equation and solve for one of the variables
2. Substitute what was found in (1) into the other equation
3. Solve the new equation for the remaining variable
4. Back substitute the value from (3) into the equation from (1)
5. Check solution.

Example 8. *Solve by subsitution:*

$$\begin{cases} y = -x - 1 \\ 4x - 3y = 24 \end{cases}$$

Substitute the first equation in the second since it is already solved for y. This yields:

$$\begin{aligned} &4x - 3(-x-1) = 24 \\ &4x + 3x + 3 = 24 \\ &7x + 3 = 24 \\ &7x = 21 \\ &x = 3 \end{aligned}$$

This yields $x = 3$ and substituting into $y = -x - 1$ yields $y = -3 - 1 = -4$. Therefore our potential solution is $(3, -4)$ and after checking, we see that it is indeed a solution.

Solving by Elimination (Addition):

1. Have both equations in the from $ax + by = c$
2. Multiply one or both equations by an appropriate non-zero number so the the sum of the coefficients of the x's or y's is zero
3. Add the equations from (2)
4. Solve the equation that is now in one variable
5. Back substitute the solution from (4) into one of the given equations and solve for the other variable
6. Check solution

Example 9. *Solve by elimination:*

$$\begin{cases} 3x + 2y = 48 \\ 9x - 8y = -24 \end{cases}$$

Notice that the x coefficient on the second equation is just 3 times the value of the coefficient on the first equation. We can use this!

$$\begin{cases} -3(3x + 2y = 48) \\ 9x - 8y = -24 \end{cases}$$

Now when we add the equations, the x variable will disappear since the first equation is now $-9x - 6y = -144$! Adding the two equations yields the equation in the variable y.

$$\begin{aligned} -6y - 8y &= -144 - 24 \\ -14y &= -168 \\ y &= 12 \end{aligned}$$

We now can pick either equation and substitute in $y = 12$ to find x.

$$\begin{aligned} &3x + 2(12) = 48 \\ &3x + 24 = 48 \\ &3x = 24 \\ &x = 8 \end{aligned}$$

Therefore our potential solution is $(8, 12)$ and after checking we see that it is indeed a solution.

Example 10. *Solve by elimination:*

$$\begin{cases} 7x + 2y = 5 \\ 2x + 3y = 16 \end{cases}$$

Notice that the x and y coefficients do not match nicely! One option we have is to multiply the top by 3 and the bottom by -2 (top by -3 and bottom by 2 works too). Doing so will cancel the y variables.

$$\begin{cases} 3(7x + 2y = 5) \\ -2(2x + 3y = 16) \end{cases}$$

$$\begin{cases} 21x + 6y = 15 \\ -4x - 6y = -32 \end{cases}$$

Now when we add the equations, the y variable will disappear. Adding the two equations yields the equation in the variable x.

$$\begin{aligned}21x - 4x &= 15 - 32\\ 17x &= -17\\ x &= -1\end{aligned}$$

We now can pick either equation and substitute in $x = -1$ to find x.

$$7(-1) + 2y = 5$$
$$-7 + 2y = 5$$
$$2y = 12 \; y = 6$$

Therefore our potential solution is $(-1, 6)$ and after checking we see that it is indeed a solution.

Lastly, three things can occur in regard to the number of solutions to a system of linear equations:

1. One solution- graphically two lines intersect at a point.
2. No solution- graphically the two lines are parallel.
3. Infinitely Many Solutions- graphically the two lines are the same.

The two cases we have not dealt with are easily classified. For a system with no solutions, when we use elimination to solve the system we end up with the statement zero = something not zero. For example

$$\begin{cases}4x + 6y = 12\\ 6x + 9y = 12\end{cases}$$

if we solve the system using elimination both the x and y variables will cancel (multiply top by -3 and bottom by 2), $0 = -12$ which is not true. Whenever this happens, it is no solution.

In the second scenario we end up with zero = zero after elimination. For example

$$\begin{cases}4x + 6y = 2\\ 8x + 12y = 4\end{cases}$$

of we solve the system we can multiply the top by -2 and once the equations are added both the x and y variables will cancel leaving $0 = 0$. Whenever this happens, it is infinitely many solutions. Also, note that one gets the second equation by multiplying the first by two.

Example 11. *A company produces and sells x units of a product. The revenue is the money generated from selling x units. The cost is the cost of producing x units. Typically, the revenue function is denoted $R(x)$ and the cost function is denoted $C(x)$.*

$R(x) =$ *(price per unit sold)* x

$C(x) =$ *fixed cost* $+$ *(cost per unit produced)* x

A company is manufacturing bionic teddy bears. They calculated the fixed cost at $\$250,000$ and it will cost $\$200$ to produce each bear. On the other hand, each bear will be sold for $\$400$.

Looking at the information given we see that the bears are sold for $\$400$ per bear, so the revenue function is given by

$$R(x) = 400x.$$

Similarly, one sees a fixed cost of $\$250,000$ and it costs $\$200$ to build each bear, so the cost function is given by

$$C(x) = 250,000 + 200x.$$

Lastly, let us find when the company will break-even, i.e. the cost and revenue functions are equal. The easiest way to solve this is to set $R(x) = C(x)$.

$$\begin{aligned}400x &= 250,000 + 200x\\ 200x &= 250,000\\ x &= 1250\end{aligned}$$

Find $C(1250) = R(1250) = 500,000$. This means that when 1250 bears are made it costs $\$500,000$ to produce the bears and at the same time the revenue is also $\$500,000$. Therefore the company is breaking-even.

For those that are into business this is also the same as the profit being zero. Since profit is revenue minus cost.

7.6

In the final section covered in this section we will look at different types of functions besides linear functions. However, I am not focusing on scatter plots like the textbook does, but we will define them.

Data represented in a visual form as a set of points is called a scatter plot. The line that best fits the data points is called a regression line, however most scatter plots are curved.

Definition 12. *The* ***exponential function*** *base b is defined by*

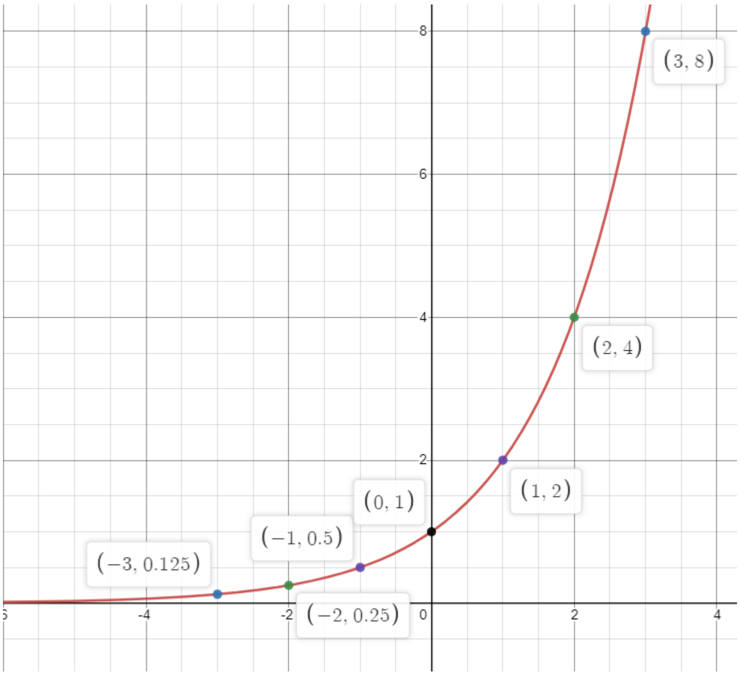
$$y = b^x$$

where b is a positive constant other than one.

This function takes x-values and sends them to b^x, i.e. $x = 2$ is sent to b^2, $x = 1/2$ is sent to $b^{1/2}$, etc. Let us graph the function $f(x) = 2^x$, the exponential function base 2 using the point plotting method. Consider the following:

x	$f(x) = 2^x$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$(-3, 1/8)$
-2	$2^{-2} = \frac{1}{4}$	$(-2, 1/4)$
-1	$2^{-2} = \frac{1}{2}$	$(-1, 1/2)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 2$	$(2, 4)$
3	$2^3 = 2$	$(3, 8)$

Plotting the points and connecting using a smooth curve yields the following graph:



The awesome thing is that every exponential graph with $b > 1$ looks exactly like this, the only thing that changes is the points.

During this section we are primarily going to look at some applications, specifically comparing two types of regressions and finding function values.

Example 13. *The world population, y, in billions, is modeled by*

$$f(x) = .074x + 2.294,$$

$$g(x) = 2.577(1.017)^x$$

where x is the number of years after 1949.

In 2000, the population was 6.1 billion, how do the models compare?

Here we need to find the x value that corresponds to the year 2000, $x = 2000 - 1949 = 51$, and plug into each of the models.

$$f(51) = .074(51) + 2.294 \approx 6.1,$$

$$g(51) = 2.577(1.017)^{51} \approx 6.1$$

In both cases the models are accurate to the year 2000. Now, which model predicts a population of 8 billion in 2026?

Similarly, we need to find the x value that corresponds to the year 2026, $x = 2026 - 1949 = 77$, and plug into each of the models.

$$f(77) = .074(77) + 2.294 \approx 8.0,$$

$$g(77) = 2.577(1.017)^{77} \approx 9.4$$

In this case, the linear model is more accurate since it predicts a world population closest to 8 billion in 2026.

For exponential functions, there is a special base called the natural base, denoted e. Where

$$e \approx 2.71828$$

Note that everything still stays the same if we use the natural base. The only thing that differs is that this base has a special button on most calculators.

Definition 14. *For $x > 0$ and $b > 0$ and $b \neq 1$, we have*

$$y = \log_b x \text{ is equivalent to } x = b^y$$

where the function $f(x) = \log_b x$ is called the ***logarithmic function*** *with base b.*

There is a relation between logarithms and exponentials where the log form is $y = \log_b x$ and the exponential form is $b^y = x$.

In essence the of the relation is that the log and exponential functions undo each other (actually are called inverses to each other). Notice the following:

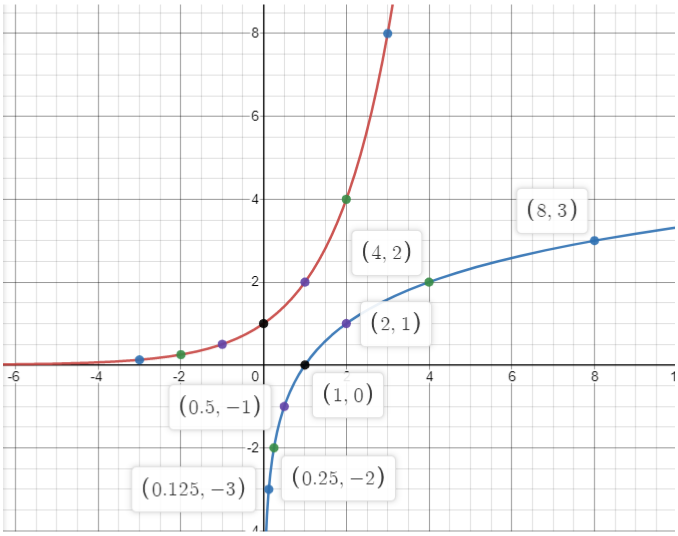
Log Form		Exponential Form
$3 = \log_2 8$	$\Leftrightarrow$	$2^3 = 8$
$2 = \log_5 25$	$\Leftrightarrow$	$5^2 = 25$
$3 = \log_4 64$	$\Leftrightarrow$	$4^3 = 64$
$-3 = \log_2 \frac{1}{8}$	$\Leftrightarrow$	$2^{-3} = \frac{1}{8}$

More interesting is that one can use the exponential graph b^x to plot the logarithmic graph $\log_b x$! If the point (x, y) is on the exponential graph then the point (y, x) is on the logarithmic graph.

All we need to do is find a few points for the graph b^x and switch the coordinates of the ordered pairs. Let us graph $g(x) = \log_2 x$ using $f(x) = 2^x$. Recall:

x	$f(x) = 2^x$	$(x, f(x))$ on 2^x	$(f(x), x)$ on $\log_2 x$
-3	$2^{-3} = \frac{1}{8}$	$(-3, 1/8)$	$(1/8, -3)$
-2	$2^{-2} = \frac{1}{4}$	$(-2, 1/4)$	$(1/4, -2)$
-1	$2^{-2} = \frac{1}{2}$	$(-1, 1/2)$	$(1/2, -1)$
0	$2^0 = 1$	$(0, 1)$	$(1, 0)$
1	$2^1 = 2$	$(1, 2)$	$(2, 1)$
2	$2^2 = 2$	$(2, 4)$	$(4, 2)$
3	$2^3 = 2$	$(3, 8)$	$(8, 3)$

Remember the x on the far left are for the exponential graph $f(x)$, the x values for the log function $g(x)$ are in the ordered pairs of the far right column. Observe that the x values for the function $g(x)$ are the function values that we found for the exponential. Plotting the two graphs side by side yields



where red is the graph $f(x) = 2^x$ and blue is $g(x) = \log_2 x$. The main thing is being able to identify the log function, and similar to the exponential all of the log functions look the same for $b > 1$.

Note that if $b = e$, then the log function is called the **natural log** function and is denoted $f(x) = \ln x$.

Example 15. *The temperature increase in a car for the first hour is modeled by*

$$f(x) = -11.6 + 13.4 \ln x$$

where x is the number of minutes.

Find the temperature increase after 50 minutes.

For this question, all we need to do is find $f(50)$ which is as follows

$$f(50) = -11.6 + 13.4 \ln(50) \approx 40.82$$

Therefore, after 50 minutes, the temperature increase in the car is 40.82 degrees.

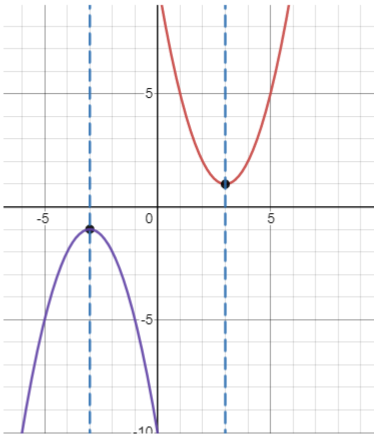
The last type of function we will look at is the quadratic functions.

Definition 16. *A **quadratic function** is any function of the form*

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers and $a =\neq 0$.

Typical quadratics are of the form



and we can use the equations to make some determinations about the graph.

1. If $a < 0$, then the graph opens down and is similar to the purple graph shown.
2. If $a > 0$, then the graph opens up and is similar to the red graph shown.

3. The points in black on each graph is called the **vertex**.
   (a) If $a < 0$, then the vertex is a maximum on the graph.
   (b) If $a > 0$, then the vertex is a minimum on the graph.
4. The vertical blue lines through the vertex is called the axis of symmetry and the graphs are symmetric with respect to the line. Meaning if you reflect one half of the graph over the line you get the complete picture!

To find the vertex, one is able to use the values a, b, c as follows, remember though that this is an ordered pair.

$$\text{Vertex: } \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

Let us create some steps to graphing quadratics:

1. Determine if the graph opens up $a > 0$ or down $a < 0$
2. Determine the coordinates of the vertex
3. Find any x-intercepts, sometimes called zeros
4. Find the y-intercept, $(0, c)$.
5. Plot the points
6. Connect with a smooth curve

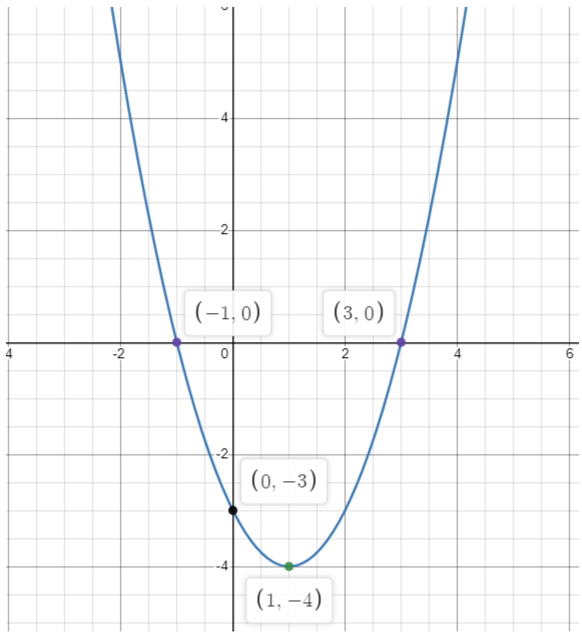
Example 17. *Graph* $f(x) = x^2 - 2x - 3$.

Let $a = 1$*,* $b = -2$*,* $c = -3$*.*

1. *Determine if the graph opens up* $a > 0$ *or down* $a < 0$

   Our graph opens up since $a = 1 > 0$
2. *Determine the coordinates of the vertex*
   (a) $\frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$
   (b) $f(1) = (1)^2 - 2(1) - 3 = -4$
   (c) So our vertex is $(1, -4)$
3. *Find any* x*-intercepts, sometimes called zeros*
   (a) There are numerous ways to do this, easiest is quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *which computes the zeros directly.*
   (b) If $a = 1$*, the other option is to ask the question what two numbers multiply to* c *and add to* b*. This produces a factorization of the function.*
   (c) $f(x) = x^2 - 2x - 3 = (x - 3)(x + 1)$ *so the zeros are* $(3, 0)$ *and* $(-1, 0)$*.*
4. *Find the* y*-intercept,* $(0, c)$*.*
   (a) The y*-intercept is* $(0, -3)$*.*
5. *Plot the points*
6. *Connect with a smooth curve*

The graph is as follows:


(−1, 0)
(3, 0)
(0, −3)
(1, −4)