Joshua Miller

jgm019@ksu.edu | 281-798-3878 | 1228 N 17th St Manhattan, KS

Research Statement

Overview: My area of interest is in Algebraic Combinatorics. I will be receiving my Ph.D. in Summer of 2021 under the guidance of Dr. Mikhail Mazin at Kansas State University (KSU). While at Sam Houston State University (SHSU), I had the opportunity to do my Master's research under Dr. Rebecca Garcia, and undergraduate research under Dr. John Alford.

Past: At Sam Houston State University, I worked with Dr. John Alford on a model of ventricular septal defects for pre-birth and post-birth infants. For my Master's research, I worked under Dr. Rebecca Garcia where we examined methods of calculating the invariant factors of the sandpile group for a family of graphs known as thick-cycle graphs. While my research on the topic did not yield a proven explicit formula, the research continued at a later MSRI session and was proven in 2016. At a later time, a second explicit formula was found for the invariant factors and a paper is in preparation on this topic.

Present: Currently, I am working with Dr. Mikhail Mazin at KSU on the topic of the generalized Pak-Stanley labeling for a family of hyperplane arrangements called multigraphical hyperplane arrangements. The first results of the project are available on arXiv [6]. The research is still ongoing and will cumulate in my doctoral dissertation, more detail will be discussed later.

In addition to my research in multigraphical hyperplanes arrangements, I am working with several colleauges on continuing our research started at the Graduate Research Workshop in Combinatorics. While at the workshop we began working on graded posets and Whitney duals.

Plans: My immediate plan for the future is to continue my research with Dr. Mazin on multigraphical hyperplane arrangements. This would be done by further generalizing our results to dimensions greater than $n = 3$. Further, I plan to continue my research on Whitney duals and graded posets with my colleagues I met at GRWC.

In the future, my research goals are two-fold. On one hand I want to start working some of the open problems in combinatorics and graph theory, but as a way to help undergraduate and graduate students start research in mathematics. The topics I am currently looking at involve edge-coloring of cubes, coverings, i.e. finding procedures for minimal coverings, and the Steiner system. For graph theory, I have a myriad of topics that would be suitable for undergraduates that involve graph colorings and the chromatic polynomials. The second aspect is personal research that I would like to begin working on, which involves topics that mirror the previous, but also looking at applications of the Tutte polynomial for hyperplane arrangements, and the Erdós-Faber-Lovász conjecture.

1.1 Whitney Duals:

A **poset** is a set of elements together with a binary relation that satisfies reflexivity, antisymmetry, and transitivity properties. For the purposes of this project, we consider finite graded posets with a minimum element $\hat{0}$. Here, graded means that every maximal chain has the same length. We say that y covers x, $x \lessdot y$, in poset P if $x < y$ and there does not exist an element $z \in P$ that satisfies $x < z < y$. Moreover, any finite poset is fully determined by the covering relations and can be represented pictorially by the Hasse diagram. See Figure 1 for an example of Hasse diagrams.

Definition 1. Given a poset P, the **Hasse diagram** is graph, $G = (V, E)$ constructed as follows. The vertex set of the diagram is the set of elements in the poset P where x is drawn lower than y if $x < y$. The edge set of the graph is given by, the edge $(x \to y) \in E$ if $x \lessdot y$.

While looking at these posets, we were interested in the Whitney numbers of the first and second kind ([8, 9, 10]) which are defined using the Möbius function

Definition 2. Let P be a locally finite poset, and consider the function $\mu : Int(P) \to \mathbb{Z}$, called the **Möbius function** of P, where the following conditions are met:

$$\mu(x,x) = 1 \text{ for all } x \in P$$
$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z), \text{ for all } x < y \text{ in } P.$$

If the poset has $\hat{0}$, then we consider $\mu(x) = \mu(\hat{0}, x)$.

It then follows the Whitney numbers are defined by

Definition 3. For a finite graded poset P with minimum element $\hat{0}$ the k^{th} Whitney number of the first kind, denoted $w_k(P)$ is defined as

$$w_k(P) = \sum_{rk(x)=k} \mu(\hat{0}, x),$$

and the Whitney number of the second kind, denoted $W_k(P)$ is defined by

$$W_k(P) = |\{x \in P : rk(x) = k\}|.$$

Where $rk : P \to \mathbb{N}$ is the rank function of P.

The way one can think of the Whitney numbers is that the Whitney numbers of the first kind sums all of the Möbius values of the elements of the poset at a rank k while the numbers of the second kind is the cardinality of the elements in P with rank k.

During our research we were interested when two posets P and Q are called Whitney duals . More precisely, we say two posets P and Q are **Whitney duals** [11] if for all $k \geq 0$ the following two conditions hold

$$|w_k(P)| = W_k(Q),$$

$$|w_k(Q)| = W_k(P).$$

During the course of the workshop we focused on tackling a series of problems:

Problem 4. *Suppose P and Q are posets that admit Whitney duals. Using poset operations, is one able to create new posets from P and Q that have a Whitney dual?*

In [12] it was shown that poset operations preserve lexicographic shellability, which is tied to edge labelings that were used in the construction of Whitney Duals in [11]. However, through our research we were able to rule out several of the poset operations which led to the following conjecture:

Conjecture 5. *Suppose P and Q are posets with Whitney duals, then* $P \times Q$ *and* $P * Q$ *have Whitney duals.*

See Figure 1 for an example of the star product. Over the course of the workshop and the months after, preliminary proofs for Conjecture 5 have been written in specific cases, however, we are still in the process of finding a generalized proof. Another two problems that interested us during the workshop was

Problem 6. *Is there a way to construct a Whitney dual of a poset without the use of labels?*

Problem 7. *How many non-isomorphic Whitney duals are there for a particular class of posets?*

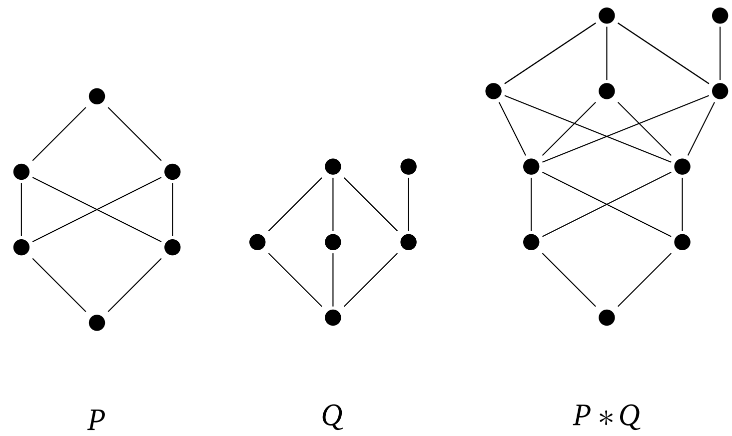

Figure 1: Both posets P and Q are Whitney self-dual posets, and on the right is the star product $P * Q$ which is also Whitney self-dual

Besides the problems listed above, I have been working on methods of creating posets that are Whitney self-dual from sequence a of integers $(m_1, \ldots, m_n)$. Moreover, I have proven the following results:

Theorem 8. *Given a sequence $(m_1, \ldots, m_n)$ of weakly decreasing non-negative integers, for $n \geq 3$ and $m_3 > 1$, then there exists a poset P with the Whitney numbers of the first kind vector*

$$\hat{w} = (1, -m_1, m_2, -m_3, \ldots, (-1)^n m_n).$$

Moreover, the poset is non-Eulerian and Whitney self-dual.

Non-Eulerian means that there exists an $x \in P$ such that $\mu(x) \neq \pm 1$. During the conference we noticed some ambiguity in the definition of graded poset, so to avoid confusion we devised system to specify what is meant. For example, P is **1-graded** if it is bounded and has a ranking function, P is **2-graded** if for every $x \in P$ the maximal intervals with x as the top element have the same length, and P is **3-graded** if it has a ranking function. The way the grading works is that 3-graded is the weakest form, while 1-graded is the strongest.

Theorem 9. *Given a sequence $(m_1, \ldots, m_n)$ of weakly decreasing non-negative integers, for $n \geq 3$ and $m_3 > 1$, then there exists a poset P with the Whitney numbers of the first kind vector*

$$\hat{w} = (1, -m_1, m_2, -m_3, \ldots, (-1)^n m_n).$$

Moreover, the poset is non-Eulerian, Whitney self-dual, and 2-graded.

1.2 Multigraphical Arrangements:

The original Pak-Stanley labeling is a bijective map from the set of regions of an extended Shi arrangement to the set of k-parking functions (see [5]). However, one can generalize this map to any finite arrangement $\mathscr{A}$ of hyperplanes in the vector space $V = \{x_1 + \cdots + x_n = 0\}$ contained in $\mathbb{R}^n$, where the hyperplanes are of the form

$$H_{ij}^a := \{x_i - x_j = a\}$$

for some positive real number a. Corresponding to this arrangement $\mathscr{A}$ is the directed multigraph $G_{\mathscr{A}}$ which is constructed on the vertex set $V = \{1, \ldots, n\}$, and the multiplicity of the edge $(i \to j)$ is given by $m_{ij} = \#\{a \in \mathbb{R}_{\geq 0} : H_{ij}^a \in \mathscr{A}\}$.

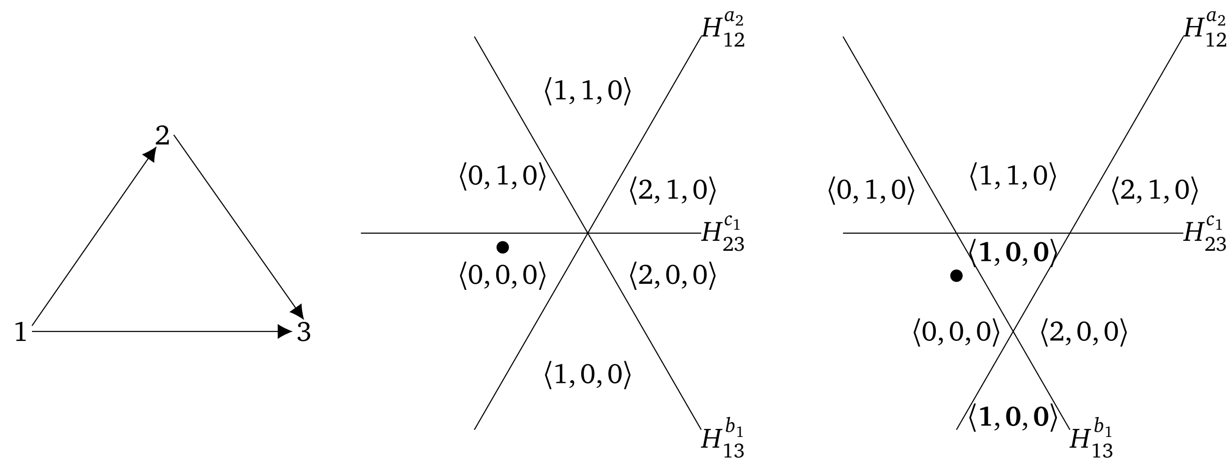

Figure 2: Two arrangements that correspond to the same digraph, but while the labelings are surjective, the one on the right is not injective. The duplicate label is in bold.

For a multigraphical arrangement, the labels that correspond to the regions of $\mathscr{A}$ are defined inductively by (i) labeling the fundamental region, i.e. the region R_0 containing the origin, the n-tuple $\lambda_{R_0} = \langle 0, \ldots, 0\rangle$, then (ii) as one crosses the hyperplane $H_{ij}^a, a > 0$, in the direction away from the origin, the ith entry of the label is increased by one. See Figure 2 for examples.

In [4] Mazin showed that the generalized Pak-Stanley labeling, a map from the regions of $\mathscr{A}$ to the set of $G_{\mathscr{A}}$-parking functions is a surjective map while also extending it to multigraphical arrangements. For k-Shi arrangements and k-parking functions, it was the case that surjectivity implied bijectivity since the set of regions and set of parking functions had the same cardinality. That is not the case for general multigraphical arrangements.

In general, the labeling is not always injective, see Figure 2 for an example. In Figure 2 we can see that the multigraph $G_{\mathscr{A}}$ does not determine the combinatorial type of the arrangement since the coefficients on the hyperplanes can be changed which in turn can create or collapse regions. This motivates the following problem:

Problem 10. *For what types of multigraphs G does there exist an arrangement $\mathscr{A}$ such that $G = G_{\mathscr{A}}$ and the generalized Pak-Stanley labeling of $\mathscr{A}$ is injective.*

The examples we considered suggested that the labeling was injective if and only if it was injective locally. Meaning

Conjecture 11. *Suppose the generalized Pak-Stanley map is not injective. Then there exists a point x such that x belongs to the boundaries of at least two distinct regions with the same label.*

Further, the examples also indicate that a stronger version of Conjecture 11 may hold, namely:

Conjecture 12. *For any multigraphical arrangement $\mathscr{A}$ and any $G_{\mathscr{A}}$-parking function λ, the closure of the union of all regions labeled by λ is connected.*

In [6], Dr. Mazin and I focused on studying the generalized Pak-Stanley labeling for central multigraphical arrangements. These arrangements $\mathscr{A}$ correspond to simple acyclic graphs and moreover, global and local injectivity is the same ([6]). We were able to prove the following result:

Theorem 13. *[6] Let G be an acyclic digraph and let $\mathscr{A}$ be the corresponding central multigraphical arrangement. Then the generalized Pak-Stanley labeling for $\mathscr{A}$ is bijective if and only if for any three vertices* i, j, k *such that* $(i \to j)$ *and* $(i \to k)$ *belong to G, either* $(j \to k)$ *or* $(k \to j)$ *also belongs to* G.

I am planning on continuing my work on generalized Pak-Stanley labelings under the supervision of Dr. Mazin with the ultimate goal of proving Conjectures 11, 12, and characterize the multigraphs admitting arrangements with bijective labelings.

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