MATH 2641: Linear Algebra I

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Instructor:	Dr. Joshua Miller	**Time:**	TTh 10:55-1:25
Review:	Exam 2	**Place:**	7/1 - 7/2

2.1

A matrix $A = [a_{i,j}]$ has main diagonal entries of the form $a_{1,1}$, $a_{2,2}$, ... and terminate depending on the values of m and n for the $m \times n$ matrix A.

The zero matrix 0 is the matrix with zeroes in every entry.

The $n \times n$ matrix I_n is the identity matrix where the main diagonal entries are all 1 and all off diagonal entries are 0.

Definition 1. *Two matrices A and B are* **equal** *if they have the same size and their corresponding columns are equal.*

Matrix Operations:
The **sum** of the matrices A and B, denoted $A + B$ is defined when A and B have the same size and each entry in $A + B$ is the sum of the corresponding entries of A and B.

Given a matrix A and scalar r, the**scalar multiple** rA is the matrix whose columns are r times the columns of A, i.e. each entry in A is multiplied by r.

Theorem 2. *Let A, B, and C be matrices of the same size and let r and s be scalars. Then*

1. $A + B = B + A$ *the sum is commutative.*
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$*, the scalar distributes.*
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Definition 3. *If A is a $m \times n$ matrix, and if B is an $n \times p$ matrix, with columns $b_1, \dots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, Ab_2, \dots, Ab_p$.*

Note: Each columns of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

Note: If A is a $m \times n$ matrix, and if B is an $n \times p$ matrix, the the product AB is defined only if the columns of A equal the rows of B, and the product AB will have size $m \times p$.

Definition 4. *If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{i,j}$ denotes the i,j- entry in AB and A is $m \times n$, then*

$$(AB)_{i,j} = a_{i,1}b_{1,i} + a_{i,2}b_{2,i} + \cdots + a_{i,n}b_{n,i}$$

Theorem 5. *Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined. Then*

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$, *watch the order when you distribute.*
3. $(B + C)A = BA + BC$, *watch the order when you distribute.*
4. $r(AB) = (rA)B = A(rB)$ *for any scalar* r.
5. $I_m A = A = AI_n$

Note: In general $AB \neq BA$

Note: Cancellation laws do not hold for matrix multiplication, i.e. if given matrices A, B, and C and $AB = AC$, we can have $B \neq C$.

Note: It is possible for two non-zero matrices to product to the zero matrix, i.e. $AB = 0$ and $A \neq 0$ and $B \neq 0$.

Definition 6. *If A is a $n \times n$ matrix and k is a positive integer, then A^k denotes the product of k copies of A.*

Definition 7. *Given a $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A.*

Theorem 8. *Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. *For any scalar* r, $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Note: The last one can be generalized to more than two matrices. The transpose of a product of matrices is equal to the product of their transposes in reverse order.

Problems to be able to solve:

1. Perform matrix addition, scalar multiplication, and matrix multiplication.
2. If given a matrix A, find A^T. Can be more than one matrix as well.
3. Determine if a product of matrices if defined.
4. Find values of k such that $AB = BA$.
5. Anything seen in lecture or practice problems.

2.2

Definition 9. *An $n \times n$ matrix A is* ***invertible*** *if there exists an $n \times n$ matrix C such that $AC = I$ and $CA = I$ where $I = I_n$. In this case, C, is an* ***inverse*** *of A and is uniquely determined by A and is denoted $C = A^{-1}$.*

Note: If the inverse exists, then there is only one.

Definition 10. *A matrix that is not invertible is called a* ***singular*** *matrix and an invertible matrix is called* ***nonsingular****.*

Theorem 11. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 12. *If A is an invertible $n \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.*

Note: The solution is unique, so there is only one solution to $A\vec{x} = \vec{b}$ if A is invertible.

Theorem 13. *Suppose A and B are invertible $n \times n$ matrices, then the following are true:*

1. $A^{-1} = (A^{-1})^{-1}$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

Note: Part 2. can be generalized as the product of invertible matrices is invertible and the inverse is the product of the inverses in reverse order.

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

Definition 14. *An* ***elementary matrix*** *is a matrix that is obtained by performing a single elementary row operation to the identity matrix.*

Note: When an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA where E is the $m \times m$ matrix created by performing the same row operation on I_m.

Note: Each elementary matrix is invertible and its inverse is the elementary matrix of the same type that transforms E back to I.

Theorem 15. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A_{-1}.*

Problems to be able to solve:

1. Find the inverse of matrices using a formula or row-reduction.
2. Use the inverse of a matrix to solve $A\vec{x} = \vec{b}$.
3. Identify elementary matrices.
4. Determine if a matrix A is singular or nonsingular.
5. Problems similar to lecture and practice problems.

2.3

Theorem 16 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. Then the following statements are equivalent:*

1. *A is an invertible matrix.*
2. *A is row equivalent to an $n \times n$ identity matrix.*
3. *A has n pivot positions.*
4. *$A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
7. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$*
9. *The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
10. *There is an $n \times n$ matrix C such that $CA = I$.*
11. *There is an $n \times n$ matrix D such that $AD = I$.*
12. *A^T is an invertible matrix.*

Note: You can replace "at least one solution" with unique solution in (7).

Note: If A and B are square matrices. If $AB = I$, then A and B are invertible and $A = B^{-1}$ and $B = A^{-1}$.

Note: For an invertible matrix A, then $A^{-1}A\vec{x} = \vec{x}$ and $AA^{-1}\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

Definition 17. *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$S(T(\vec{x})) = \vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$
$$T(S(\vec{x})) = \vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n.$$

*We say that S is the **inverse** of T.*

Theorem 18. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In this case, the linear transformation S given by $S(x) = A^{-1}\vec{x}$ is the unique function satisfying*

$$S(T(\vec{x})) = \vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$

$$T(S(\vec{x})) = \vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n.$$

Problems to be able to solve:

1. Use the theorem to determine if a matrix is invertible and justify it.
2. Find the inverse of a matrix transformation.
3. Conceptual problems similar to MLM.

2.4

Definition 19. *A* ***partitioned matrix*** *is a matrix that has been constructed from other smaller matrices, these matrices are also called* ***block matrices*** *and the entries of a partitioned matrix are called blocks or submatices.*

Note: Given two matrices A and B that are the same size and partitioned in the same way, then the sum $A+B$ is just the sum of the corresponding blocks of A and B.

Note: For scalar multiplication, the multiplication is done block by block.

Theorem 20. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix, then*

$$AB = [col_1(A) \ \cdots \ col_n(A)] \begin{bmatrix} row_1(B) \\ \vdots \\ row_n(B) \end{bmatrix} = col_1(A)row_1(B) + \cdots + \ col_n(A)row_n(B)$$

Definition 21. *A* ***block diagonal matrix*** *is a partitioned matrix with zero blocks off the main diagonal of blocks.*

Note: A block diagonal matrixi is invertible if each block on the diagonal is invertible.

Problems to be able to solve:

1. Problems similar to the lecture, finding the inverse of a partitioned matrix.
2. Problems similar to practice problems, finding formulas for blocks in terms of other blocks.

2.8

Definition 22. *A* ***subspace*** *of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:*

1. *The zero vector is in H.*
2. *For each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u} + \vec{v}$ is in H.*
3. *For each $\vec{u}$ in H and each scalar c, the vector $c\vec{u}$ is in H.*

Note: Parts (2) and (3) state that a subspace H is closed under addition and scalar multiplication.

Definition 23. *For $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_p$ is a subspace of $\mathbb{R}^n$. We will now call $Span\{\vec{v}_1, \ldots, \vec{v}_p\}$ the* ***subspace spanned by*** *$\vec{v}_1, \ldots, \vec{v}_p$.*

Definition 24. *The subspace consisting of only the zero vector is called the* ***zero subspace****.*

Definition 25. *The* ***column space*** *of a matrix A is the set $Col\ A$ of all linear combinations of the columns of A.*

Note:The column space of a $m \times n$ matrix is $\mathbb{R}^m$.

Note: The column space is $\mathbb{R}^m$ if and only if the pivots in all columns of A (columns span R^m). Otherwise it is just a subspace of $\mathbb{R}^m$.

Note: The column space of A is the set of all vectors $\vec{b}$ where the equation $A\vec{x} = \vec{b}$ has a solution, i.e. $\vec{b}$ can be written as a linear combination of the columns of A.

Definition 26. *The **null space** of a matrix A is a the set Nul A of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$.*

Note: When a matrix A has n columns, then the solutions to the homogeneous equation belongs to $\mathbb{R}^n$ and the null space is a subspace of $\mathbb{R}^n$.

Theorem 27. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\vec{x} = \vec{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Definition 28. *A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.*

Note: The set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the **standard basis** for $\mathbb{R}^n$.

Note: To find the basis for Nul A, solve the homogeneous equation and write the solution in its parametric form. The vectors created in the parametric form are the basis vectors for the null space, i.e. if you have k free variables, then you have k basis vectors.

Theorem 29. *The pivot columns of a matrix A form a basis for the column space of A.*

Note: Do row and column operations to change A into a REF form to find the pivot columns, but use the columns of the original matrix A to create the basis.

Note: The smallest set of vectors that spans a subspace is actually linearly independent.

Problems to be able to solve:

1. Determine if something is a subspace.
2. Determine if a vector is in the column space of a matrix A.
3. Determine if a set of vectors is a basis.
4. Determine the basis for the column space and null space.
5. Problems similar to lecture and practice problems.

2.9

Note: If we have a basis for a subspace H, then there is exactly one way to write a vector in H as a linear combination of basis vectors.

Definition 30. *Suppose the set $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis for a subspace H. For each vector $\vec{x}$ in H, the **coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$** are the weights $c_1, \dots, c_p$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$ and the vector in $\mathbb{R}^p$*

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

*is called the **coordinate vector of $\vec{x}$ relative to the basis $\mathcal{B}$**.*

Note: If a subspace of H has a basis of p vectors, then every basis of H must consist of exactly p vectors.

Definition 31. *The* ***dimensions*** *of a nonzero subspace H, denoted dim H is the number of vectors in a basis for H. The dimension of the zero subspace $\{\vec{0}\}$ is defined to be 0.*

Note: The space $\mathbb{R}^n$ has dimension n and thus every basis of $\mathbb{R}^n$ has n vectors.

Note: The dimension of the null space is the number of free variables for the homogeneous equation.

Definition 32. *The* ***rank*** *of matrix A, denoted rank A, is the dimension of the column space of A.*

If a matrix A has n pivot columns, then rank A is n.

Theorem 33 (Rank Theorem)**.** *If a matrix A has n columns, then rank A + dim Nul $A = n$.*

Theorem 34 (Basis Theorem)**.** *Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.*

Theorem 35 (Invertible Matrix Theorem Continued)**.** *Let A be an $n \times n$ matrix, then the following statements are equivalent to the statements from 2.3*

13. *Columns of A form a basis of $\mathbb{R}^n$.*
14. *Col $A = \mathbb{R}^n$.*
15. *rank $A = n$.*
16. *dim Nul $A = 0$.*
17. *Nul $A = \{\vec{0}\}$.*

Problems to be able to solve:

1. Determine the rank and dimension of spaces.
2. Use the new parts to the Inverse Matrix Theorem.
3. Problems similar to MLM homework.

3.1

Definition 36. *The* ***determinant*** *of a square matrix is a numerical value that is calculated using the entries of a matrix according to a certain formula.*

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with the plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. Symbolically

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} \det A_{1j}.$$

Note: Remember that A_{ij} is the matrix obtained from A by removing the i^{th} row and j^{th} column.

Definition 37. *The* (i, j)*-**cofactor** of* A *is the number* C_{ij} *where*

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Note: That the cofactor has the sign built into the defintion.

Theorem 38. *The determinant of a* $n \times n$ *matrix* A *can be computed by cofactor expansion across any row or down any column.*

$$\textit{Across row } i\textit{:} \ \det A = a_{i1}Ci1 + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$\textit{Down column } j\textit{:} \ \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Note: We want to expand along a row or column with the most zeroes.

Theorem 39. *If* A *is a triangular matrix (upper or lower), then* $\det A$ *is the product of the entries on the main diagonal of* A*.*

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

Problems to be able to solve:

1. Find the determinant of a matrix.
2. Find the cofactor expansion of a matrix A if a row or column is specified.
3. Use cofactor expansion to find the determinant.
4. Find the determinant of a triangular matrix.
5. Problems similar to the homework.

3.2

Theorem 40. *Let* A *be a square matrix, then the following are true*

1. *If a multiple of one row of* A *is added to another row to produce a matrix* B*, then* $\det A = \det B$*.*
2. *If two rows of* A *are interchanged to produce* B*, then* $\det B = -\det A$*.*
3. *If one row of* A *is multiplied by* k *to produce* B*, then* $\det B = k \det A$*.*

Note: The word row can be replaced with column and this is still a true theorem. We just don't discuss column operations.

Theorem 41. *Suppose a matrix* A *is reduced to the echelon form* U*, using row replacements and* r *row interchanges, then*

$$\textit{If } A \textit{ is invertible:} \ \det A = (-1)^r (\textit{product of the pivots of } U)$$

$$\textit{If } A \textit{ is not invertible:} \ \det A = 0$$

Note: Since we are dealing with a square matrix, every echelon form is an upper triangular matrix, however the entries on the main diagonal can be zero which gives us the two cases here.

Note: You can also exchange the word row with column and it be true for the previous theorem.

Theorem 42. *A square matrix* A *is invertible if and only if* $\det A \neq 0$*.*

Theorem 43. *If* A *is an* $n \times n$ *matrix, then* $\det A^T = \det A$*.*

Theorem 44. *If A and B are $n \times n$ matrices, then* $\det AB = (\det A)(\det B)$

Note: This can generalize to more than two matrices.

Theorem 45. *Given a square matrix A with block diagonal entries $A_1, \ldots A_k$, and zeroes everywhere else, then*

$$\det A = (\det A_1)(\det A_2) \cdots (\det A_k)$$

Problems to be able to solve:

1. Use row operations to calculate determinants.
2. Determine if a matrix is invertible using determinants.
3. Find the determinant of a product of matrices.
4. Determine if a set of vectors is linearly independent using determinants.
5. Problems from MLM.

Theorem 46 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. Then the following statements are equivalent:*

1. *A is an invertible matrix.*
2. *A is row equivalent to an $n \times n$ identity matrix.*
3. *A has n pivot positions.*
4. *$A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
7. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$*
9. *The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
10. *There is an $n \times n$ matrix C such that $CA = I$.*
11. *There is an $n \times n$ matrix D such that $AD = I$.*
12. *A^T is an invertible matrix.*
13. *Columns of A form a basis of $\mathbb{R}^n$.*
14. *Col $A = \mathbb{R}^n$.*
15. *rank $A = n$.*
16. *dim Nul $A = 0$.*
17. *Nul $A = \{\vec{0}\}$.*
18. $\det A \neq 0$.