Study Guide for Exam 2 in MATH 2420

Section 3.1

Definitions, Theorems, etc from the section:

1. A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate is the set of all values that can be substituted for the variable.

2. If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements in D that make $P(x)$ true, denoted $\{x \in D | P(x)\}$.

3. The symbol $\forall$ is called the **universal quantifier** had represents the phrases, for every, for each, for any, given any, and for all.

4. Let $Q(x)$ be a predicate with D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$". It is defined to be true if and only if it is true **FOR EVERY** x in the domain. It is false if and only if $Q(x)$ is false **FOR AT LEAST ONE** x in the domain.

5. Any value x in the domain D of a universal statement that makes $Q(x)$ false is called a **counterexample**.

6. One way to check if a universal statement is true or false is to use the **method of exhaustion** which means that you check every value of the domain to see if the predicate is true or false.

7. The symbol $\exists$ is called the **existential quantifier** and represents the phrase there exists.

8. Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D,$ such that $Q(x)$." It is true if and only if $Q(x)$ is true **FOR AT LEAST ONE** x in D. It is false if and only if $Q(x)$ is false **FOR ALL** x in D.

9. Formal language means we use the quantifier symbols, set symbols, variables, and etc in our formulation. For example "$\forall$ people x, x is tall" or $\exists x \in D,$ s.t.$x^2 = 1$" are both examples of formal of differing degrees.

10. Informal languages uses no symbols, variables or anything of the sort. It is purely done through words, for example "all dogs are cats" or " there exists a triangle such that the triangle has three ninety degree angles."

11. The **universal conditional statement** is a statement of the form "$\forall x$ if $P(x)$ then $Q(x)$. This statement can also be written as $\forall x \in U, Q(x)$ where U is a subset of D were $P(x)$ is true.

12. The statement "$\exists x$ such that $P(x)$ and $Q(x)$ can be rewritten as "$\exists x \in D$ such that $Q(x)$ where D is the set of values of x where $P(x)$ is true.

13. Let $P(x)$ and $Q(x)$ be predicates with the common domain of x D.

    (a) The notation $P(x) \rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$.

    (b) The notation $P(x) \leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have the same truth set.

Problems to be able to Solve:

1. Determine if a statement is true or false.
2. Find a counterexample to a statement.
3. Write a statement in an informal or formal way.
4. Rewrite a statement in a way that is stated.
5. State the hypothesis and conclusion.
6. Determine the truth set for a predicate.
7. Any example covered in lecture, homework, or in the in-class assignments.

Section 3.2

Definitions, Theorems, etc from the section:

1. The **negation of a universal statement** ("$\forall x \in D, Q(x)$) is "$\exists x \in D$ such that $\sim Q(x)$."
2. The negation of a universal statement "all are" is logically equivalent to an existential statement "some are not" or "there is at least one that is not."
3. The **negation of an existential statement** ("$\exists x \in D$, such that $Q(x)$") is "$\forall x$ in D, $\sim Q(x)$."
4. The negation of a existential statement "some are" is logically equivalent ot a universal statement " none are" or " all are not."
5. The **negation of the universal conditional** ("$\forall x$, if $P(x)$ then $Q(x)$") is "$\exists x$ such that $P(x)$ and $\sim Q(x)$."

6. If $Q(x)$ is a predicate and the domain of $D = \{x_1, \ldots, x_n\}$ then the statement $\forall x \in D, Q(x)$ is equivalent to
$$Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n).$$
7. If $Q(x)$ is a predicate and the domain of $D = \{x_1, \ldots, x_n\}$ then the statement $\exists x \in D,$ such that $Q(x)$ is equivalent to
$$Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n).$$
8. Given a statement of the form $\forall x \in D$ if $P(x)$ then $Q(x)$, the statement is called **vacuously true** if and only if the statement $P(x)$ is false for all $x \in D$.
9. Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.
   (a) Its **contrapositive** is $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
   (b) Its **converse** is $\forall x \in D$, if $Q(x)$ then $P(x)$.
   (c) Its **inverse** is $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.
10. The universal conditional is logically equivalent to its contrapositive.
11. $\forall x, R(x)$ is a **sufficient condition** for $S(x)$ means: $\forall x$, if $R(x)$ then $S(x)$.
12. $\forall x, R(x)$ is a **necessary condition** for $S(x)$ means: $\forall x$, if $\sim R(x)$ then $\sim S(x)$ or $\forall x$, if $S(x)$ then $R(x)$.
13. $\forall x, R(x)$ **only if** $S(x)$ means: $\forall x$, if $\sim S(x)$ then $\sim R(x)$ or $\forall x$, if $R(x)$ then $S(x)$
14. Some other ways we can write informal quantified statements:

Original	Alternate
All A are B.	There are no A that are not B.
Some A are B.	There exists at least one A that is B.
No A are B.	All A are not B.
Some A are not B.	Not all A are B.

15. Some negations we talked about in class that are informal:

	Negation	
All A are B	$\Leftrightarrow$	Some A are not B
Some A are B	$\Leftrightarrow$	No A are B.

Problems to be able to Solve:

1. Be able to negate a statement in an informal or formal way.
2. Be able to rewrite a statement in an informal way.

3. Determine if a statement or its negation is true or false.
4. Rewrite statements in an if-then form.
5. Rewrite statemens in their contrapositive, inverse, or converse forms.
6. Any example covered in lecture, homework, on in the in-class assignments.

Section 3.3

Definitions, Theorems, etc from the section:

1. Whenever a statement has more than one quantifier, we treat the actions of the quantifiers in the order in which the quantifiers occur in the statement.
2. "$\forall x \in D, \exists y \in E$ such that x and y satisfies $P(x, y)$," this means that for each x I can find a y such that $P(x, y)$ is true. You are allowed to find a different value of y for each x.
3. "$\exists x \in D$, such that $\forall y \in E$ x and y satisfies $P(x, y)$," this means that there is a x where regardless of the y chosen, $P(x, y)$ is true. This x must work with every y.
4. The **reciprocal** of a real number a is a real number b such that $ab = 1$.
5. The negation of the statement '$\forall x \in D, \exists y \in E$ such that x and y satisfies $P(x, y)$," is '$\exists x \in D$, such that $\forall y \in E$, $\sim P(x, y)$."
6. The negation of the statement "$\exists x \in D$, such that $\forall y \in E$ x and y satisfies $P(x, y)$," is $\forall x \in D, \exists y \in E$ such that $\sim P(x, y)$.
7. Remember that a statement containing both $\forall$ and $\exists$, the order matters. If you change the order it can change the meaning of the statement.
8. Remember that statement that contains multiple $\forall$ or $\exists$ (not both, multiples of one type) then the order does not affect the meaning.

Problems to be able to Solve:

1. Determine if a statement of the form $P(x, y)$ is true or false for a given x, y.
2. Find values that make a predicate true.
3. Answer questions related to a Tarksi world.
4. Rewrite formal and informal statements.
5. Any example covered in lecture, homework, on in the in-class assignments.

Section 3.4

Definitions, Theorems, etc from the section:

1. **Universal Instatiation** means that if a property is true of everything in a set, then it is true for any particular element in the set.
2. **Universal Modus Ponens**,is a valid argument form of the form:

   $\forall x$ if $P(x)$ then $Q(x)$

   $P(a)$ for a particular a

   $\therefore Q(a)$
3. **Universal Modus Tollens**, is a valid argument form of the form:

   $\forall x$, if $P(x)$ then $Q(x)$

   $\sim Q(a)$ for a particular a

   $\therefore \sim P(a)$
4. To say that an argument form is **valid** means that no matter what particular predicates are substituted for the predicate symbols in its premises, if the premises are true then the conclusion is also true. An argument is valid if and only if the argument form is valid. It is called **sound** if and only if its form is valid and its premises are true.
5. **Universal Transitivity**, we have a string of if- then statements where one implies the second, the second implies the third and so on, therefore the first implies the last statement:

   $\forall x$ if $P(x) \to Q(x)$

   $\forall x$ if $Q(x) \to R(x)$

   $\therefore \forall x$ if $P(x) \to R(x)$
6. **Converse Error, Quantified**

   $\forall x$ if $P(x) \to Q(x)$

   $Q(a)$ for a particular a.

   $\therefore P(a)$
7. **Inverse Error, Quantified**

   $\forall x$ if $P(x) \to Q(x)$

   $\sim P(a)$ for a particular a

   $\therefore \sim Q(a)$

Problems to be able to Solve:

1. Determine the argument form for a given argument.
2. Be able to fill in the missing parts of a given argument.
3. Identify Converse/Inverse errors.
4. Use venn diagrams to show that a statement is true or false.
5. Any example covered in lecture, homework, on in the in-class assignments.

Section 4.1

Definitions, Theorems, etc from the section:

1. The set of integers is closed under addition, subtraction, and multiplication. Meaning that if $a, b \in \mathbb{Z}$, then $a+b, a-b, ab \in \mathbb{Z}$.
2. An integer n is **even** if and only if n equals twice some integer, i.e. $n = 2s$ for some integer s.
3. An integer n is **odd** if and only if n equals twice some integer plus one, i.e. $n = 2r + 1$ for some integer r.
4. An integer n is prime if and only if $n > 1$ and for all positive integers r and s, if $n = rs$, then either $r = 1$, $s = n$ or $s = 1$, $r = n$.
5. An integer n is **composite** if and only if $n > 1$ and $n = rs$ for some integers r, s with $1 < r < n$ and $1 < s < n$.
6. To prove an existential statement, $\exists x \in D$, such that $Q(x)$ we can use **constructive proofs of existence** which either require us to produce at least one x such that $Q(x)$ is true or give directions on how to find a x.
7. To disprove by counterexample, for the statement $\forall x \in D$, if $P(x)$ then $Q(x)$, we need to find at least one value x such that $P(x)$ is true and $Q(x)$ is false. This x is called a **counterexample**.
8. To prove a statement of the form $\forall x \in D$, if $P(x)$, then $Q(x)$, is to use the **method of exhaustion** which requires us to check the statement for every value in the domain. This is particularly helpful in cases when the domain is finite.
9. **Generalizing from the Generic Particular:** To show that every element of a set satisfies a certain property, suppose x is a particular but arbitrarily chosen element of the set, and show that x satisfies the property.

10. **Method of Direct Proofs:**
    (a) Express the statement to be proved in the form "For every $x \in D$, if $P(x)$ then $Q(x)$.
    (b) Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true.
    (c) Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules for logical inference.

Problems to be able to Solve:

1. Use definitions to justify why a statement is true or false (for even/odd/prime/composite).
2. Prove existential statements.
3. Disprove universal statements by finding counterexamples.
4. State the starting point and conclusion of a proof.
5. Write a proof involving even, odd, prime, or composite numbers similar to in-class assignments.
6. Any example covered in lecture, homework, on in the in-class assignments.

Section 4.2

Definitions, Theorems, etc from the section:

1. **Directions for Writing Proofs for Universal Statements:**
    (a) Copy the statement of the theorem to be proven on your paper.
    (b) Mark the beginning of the proof with the word proof.
    (c) Make your proof self-contained, meaning that you label every variable in your proof. item Write in complete, grammatically correct sentences.
    (d) Keep your reader informed about the status of each statement in your proof. State whether it is assumed, proven, or still needs to be proved.
    (e) Give a reason for each assertion in your proof. Every statement should come from a hypothesis, a definition of a term in the theorem, or the result of something obtained earlier in the proof (or from a different proof).
    (f) Make sure each sentence follows from the previous statements. Make sure that each part of the argument is related to the previous sentence or a combination of the previous statements.

(g) Display any equations or inequalities.

2. **Avoid Common Mistakes:**
   (a) Don't argue from examples, use them to generalize if possible. This is because sometimes things only hold for specific values or elements. For example $a^2 + b^2 = (a+b)^2$ is false in general, but if $a = 0$ then it is true.
   (b) Don't use the same letter for two different meanings.
   (c) Don't jump to conclusions.
   (d) Don't assume what you are try to prove.
   (e) Don't confuse what is known and what needs to be shown.
   (f) Be careful with the word "any" when the correct word is "some." For example, $n = 2s$ for some integer s is different that $n = 2s$ for any integer s.
   (g) Be careful with misuse of the word "if" especially when the word "because" is what is meant to be used. For example, suppose p is a prime number. If p is prime, then p cannot be written as the product of two smaller positive integers. Here the if should be the word because.
3. Remember that to disprove an existential statement, we have to show that it is false for every x in the domain.

Problems to be able to Solve:

1. Use definitions to justify why a statement is true or false (for even/odd/prime/composite).
2. Prove or disprove existential statements.
3. Prove or disprove universal statements by finding counterexamples.
4. State the starting point and conclusion of a proof.
5. Write a proof involving even, odd, prime, or composite numbers similar to in-class assignments.
6. Any example covered in lecture, homework, on in the in-class assignments.