MATH 2641: Linear Algebra I

Summer 2022 CRN: 55755-010

Instructor:	Dr. Joshua Miller	**Time:**	TTh 10:55-1:25
Review:	Exam 3	**Place:**	7/22 - 7/23

4.1

Definition 1. *A **vector space** is a nonempty set V of objects called vectors together with two operations called addition and multiplication by scalars. The set V satisfies the following:*

1. *The sum of $\vec{u}$ and $\vec{v}$, denoted $\vec{u}+\vec{v}$ is in V.*
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.
3. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$.
4. *There is a zero vector, $\vec{0}$ in V such that $\vec{u}+\vec{0}=\vec{u}$.*
5. *For each $\vec{u}$ in B, there is a vector $-\vec{u}$ in V such that $\vec{u}+(-\vec{u}=\vec{0}$.*
6. *The scalar multiple of $\vec{u}$ by c, denoted $c\vec{u}$ in V.*
7. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$.
8. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$.
9. $c(d\vec{u})=(cd)\vec{u}$.
10. $1\vec{u}=\vec{u}$.

Note: The set of matrices is not a vector space.

Note: The set of matrices of a fixed size is a vector space.

Note: The set of invertible matrices is not a vector space under normal addition.

Note: The set of polynomials of degree at most n is a vector space.

Definition 2. *A **subspace** of a vector space V is a subset H of V that satsifies:*

1. *the zero vector of V is in H.*
2. *H is closed under vector addition.*
3. *H is closed under scalar multiplication.*

If H satisfies these three properties, then H is a subspace.

Theorem 3. *If $\vec{v}_1,\dots,\vec{v}_p$ are in a vector space V, then span$\{\vec{v}_1,\dots,\vec{v}_p\}$ is a s subspace of V.*

Note: This is the subspace spanned by the vectors $\vec{v}_1,\dots,\vec{v}_p$.

Problems to be able to solve:

1. Determine if a set is a subspace, vector space, or neither.
2. Find spanning sets for a space.
3. Solve problems similar to lecture.

4.2

Definition 4. *The **null space** of an $m \times n$ matrix A, denoted Nul A, is the set of all solutions to the homogenous equation.*

$$Nul\ A = \{x : x \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\}.$$

Theorem 5. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

Note: The null space is defined implicitly, and to explicitly state the null space we have to solve the homogeneous system.
Note: The spanning set of Nul A found in class is automatically linearly independent.

Note: The number of vectors in the spanning set is the number of free variables to the homogeneous equation.

Definition 6. *The **column space** of an $m \times n$ matrix A, denoted Col A , is the set of all linear combinations of the columns of A.*

$$Col\ A = \ Span\{\vec{a}_1, \dots, \vec{a}_n\} = \{\vec{b} : \vec{b} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}.$$

Theorem 7. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Definition 8. *The **row space** of a $m \times n$ matrix A, denoted Row A, is the set of all linear combinations of the row vectors of the matrix A.*

Theorem 9. *The row space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

Note Row A = Col A^T.

Definition 10. *A **linear transformation** from a vector space V into a vector space W is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W such that*

1. *$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in V.*
2. *$T(c\vec{u}) = cT(\vec{u})$ for all scalars c, and vectors $\vec{u}$ in V.*

Definition 11. *The **kernel**, i.e. null space, of T is the set of vectors $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$.*

Definition 12. *The **range** of T is the set of all vectors in W of the form $T(\vec{u})$ where $\vec{u}$ is in V.*

Note: These are just the null space and column space of A respectively. Further the range of T is a subspace of W and the kernel is a subspace of V.

Problems to be able to solve:

1. Determine if a vector is in the row, column or null space.
2. Find a vector in the row, column or null space.
3. Find the spanning set of the null space.
4. Solve problems similar to lecture.

4.3

Definition 13. *A set of vectors* $\{\vec{v}_1, \dots, \vec{v}_p\}$ *in a vector space* V *is said to be* ***linearly independent*** *if the vector equation*

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution. Moreover, the set is said to be ***linearly dependent*** *if there exists weights* $c_1, \dots, c_p$ *not all 0 zero such that*

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}.$$

Theorem 14. *A set containing the zero vector is linearly dependent.*

Theorem 15. *A set of two vectors is linearly dependent if and only if one is a multiple of the other.*

Theorem 16. *A set consisting of a single non-zero vector is linearly dependent.*

Theorem 17. *An indexed set* $\{\vec{v}_1, \dots, \vec{v}_p\}$ *of two or more vectors with* $\vec{v}_1 \neq 0$*, is linearly dependent if and only if some vector* $\vec{v}_j$ *is a linear combination of the preceding vectors* $\vec{v}_1, \dots, \vec{v}_{j-1}$*.*

Definition 18. *Let* H *be a subspace of a vector space* V*. A set of vectors* $\mathcal{B}$ *in* V *is a* ***basis*** *for* H *if* $\mathcal{B}$ *is a linearly independent set and the subspace spanned by* $\mathcal{B}$ *is* H*.*

Note: For the null space, the number of basis vectors is the number of free variables for the homogeneous equation.

Theorem 19 (Spanning Set)**.** *Let* $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ *be a set in a vector space* V*, and let* H*=Span*$\{\vec{v}_1, \dots, \vec{v}_p\}$ *.*

1. *If one of the vectors in* S *is a linear combination of the remaining vectors in* S*, say* $\vec{v}_i$*, then the set formed from* S *by removing* $\vec{v}_i$ *still spans* H*.*
2. *If* $H \neq \{\vec{0}\}$*, some subset of* S *is a basis for* H*.*

Note: For the column space, the number of basis vectors is the number of pivot columns in our matrix. The columns of the original matrix are the basis vectors for the space.

Note: For the row space, the number of basis vectors is the number of pivot rows in our matrix. The rows corresponding to the pivot rows form the basis of the row space.

Theorem 20. *The pivot columns of a matrix* A *form a basis for Col* A*.*

Theorem 21. *The elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.*

Theorem 22. *If two matrices* A *and* B *are row equivalent, then their row spaces are the same. Further, if* B *is the echelon form, then a basis for the row space is the non-zero rows of* B *and this is also a basis for the row space of* A *as well.*

Problems to be able to solve:

1. Determine if a set is a spanning set, linearly independent, or a basis.
2. Find the basis of the null space, column space, or row space.
3. Find the basis for the span of a set of vectors.
4. Find the basis for the subspace of a polynomial space or matrix space.
5. Solve problems similar to lecture.

4.4

Theorem 23 (Unique Representation)**.** *Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n$ be a basis for a vector space V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that*

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n.$$

Definition 24. *Suppose the set $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ is a basis for a subspace H. For each vector $\vec{x}$ in H, the **coordinates of $\vec{x}$ relative to the basis** $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$ and the vector in $\mathbb{R}^p$*

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

*is called the **coordinate vector of $\vec{x}$ relative to the basis** $\mathcal{B}$. The mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is the **coordinate mapping** determined by $\mathcal{B}$.*

Note: Changing basis is like changing the grid paper we use to plot.

Definition 25. *For a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$, let $P_{\mathcal{B}} = [\vec{b}_1, \ldots, \vec{b}_n]$ and for some $\vec{x}$ in a vector space V*

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

*then $P_{\mathcal{B}}$ is the **change of coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Moreover, $P_{\mathcal{B}}^{-1}$ is the change of coordinates matrix from the standard basis in $\mathbb{R}^n$ to the basis $\mathcal{B}$. Meaning*

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

Theorem 26. *Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V on to $\mathbb{R}^n$.*

Note: $\mathbb{R}^n$ is isomorphic to $\mathbb{P}_{n-1}$.

Note: Isomorphic means that every vector space calculation in one vector space is accurately reproduced in the other. We can think of the spaces as being "the same."

Theorem 27. *Assume that $\mathcal{B}$ is a basis for a vector space V. A set $\{\vec{u}_1, \ldots, \vec{u}_n\}$ in V is linearly independent if and only if $\{[\vec{u}_1]_{\mathcal{B}}, \ldots, [\vec{u}_n]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.*

Problems to be able to solve:

1. Find the coordinate vectors for different bases.
2. Find the change of coordinate matrix.
3. Use coordinate vectors to determine if a set of vectors or polynomials is linearly independent.
4. Solve problems similar to lecture.

4.5

Theorem 28. *If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$, then any set in V containing more than n vectors is linearly dependent.*

Theorem 29. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

Definition 30. *If a vector space V is spanned by a finite set, then V is called **finite-dimensional** and the dimension of V, written $\dim V$, is the number of vectors in a basis. The dimension of the zero vector space is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.*

Theorem 31. *Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to a basis of H. Further, if H is finite dimensional then*

$$\dim H \leq \dim V.$$

Theorem 32 (Basis Theorem)**.** *Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is also a basis for V.*

Definition 33. *The **rank** of a $m \times n$ matrix A is the dimension of the column space. The **nullity** of A is the dimension of the null space of A.*

Note: $rankA = \text{dim Col } A = \text{dim Row } A =$ number of pivot columns or pivot rows.

Note: Nullity of A is the number of free variables in the homogeneous equation.

Theorem 34 (Rank)**.** *For a matrix A of size $m \times n$, the rank and nullity satisfy*

$$Rank\ A\ +\ Nullity\ A\ = n$$

Note: Since Row $A = $ Col A^T, then rank $A = $ rank A^T.

Note: If A is $m \times n$, then a pivot in every row means that $A\vec{x} = \vec{b}$ has at least one solution, each $\vec{b}$ in $\mathbb{R}^m$ is in the span of the columns of A, the columns of A span $\mathbb{R}^m$, and there is a pivot in every row.

Note: If A is $m \times n$, then a pivot in every column means there are no free variables, $A\vec{x} = \vec{b}$ has at most one solution, the dimension of Col A is n, and the columns of A are linearly independent.

Note: If A is a square matrix, then a pivot in every row is the same as every column, so the the equation $A\vec{x} = \vec{b}$ has a unique solution.

Problems to be able to solve:

1. Find the dimension of a subspace.
2. Find the dimension of null, column, and row space.
3. Use the Rank-Nullity theorem to answer questions.
4. Solve problems similar to lecture.

4.6

Theorem 35. *Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$ be two bases of a vector space V. Then there is a unique $n \times n$ matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that*

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$, i.e.

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}}].$$

Note: The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is called the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ and converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

Note: The columns of the change of coordinate matrix are linearly independent since they are the coordinate vectors of a linearly independent set.

Note: Since the matrix is square, it is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

Note: If $\mathcal{C}$ is the standard basis (any of them) then the change of coordinates matrix is the previously define $P_{\mathcal{B}}$ from earlier.

Problems to be able to solve:

1. Map a coordinate vector in one base to a coordinate vector in another.
2. Find the change of coordinates matrix between two nonstandard bases.
3. Application to polynomial vector spaces.
4. Solve problems similar to lecture.

5.1

Definition 36. *An **eigenvector** of an $n \times n$ matrix A is a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is a called an **eigenvalue** of A if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$; such an $\vec{x}$ is called an eigenvector corresponding to λ.*

Note: There are an infinite number of eigenvectors that correspond to a eigenvalue, we just pick one.

Definition 37. *The set of all solutions to $(A - \lambda I)\vec{x} = 0$ is called the **eigenspace** of A corresponding to λ.*

Note: Solving $(A - \lambda I)\vec{x} = 0$ if equivalent to finding nul $(A - \lambda I)$.

Note: If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

Theorem 38. *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Theorem 39. *If $\{\vec{v}_1, \ldots, \vec{v}_r\}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.*

Note: To check if a scalar is an eigenvalue, say λ, must check $(A - \lambda I)\vec{x} = 0$ has a nontrivial solution.

Note: To find the eigenspace or eigenspace basis for $\lambda = \alpha$, find the parametric solution to $(A - \alpha I)\vec{x} = 0$.

Problems to be able to solve:

1. Determine if a vector or scalar is an eigenvector or eigenvalue.
2. Find eigenvalues of matrices.
3. Find a basis for an eigenspace.
4. Solve problems similar to lecture.

5.2

Definition 40. *The* ***characteristic polynomial*** *of a matrix $n \times n$ matrix A is a polynomial of degree n defined by*

$$\det(A - \lambda I).$$

The ***characteristic equation*** *is defined by*

$$\det A - \lambda I)\vec{x} = 0.$$

Last a scalar λ is a eigenvalue of A if it satisfies the characteristic equation, i.e. is a root of the characteristic polynomial.

Note: If 0 is an eigenvalue for A, then A is not invertible.

Note: The **algebraic multiplicity** of an eigenvalue is its multiplicity as root of the characteristic polynomial. The **geometric multiplicity** of an eigenvalue is the dimension of the corresponding eigenspace.

Definition 41. *For $n \times n$ matrices A and B, we say that A is* ***similar*** *to B if there is an invertible matrix P such that*

$$A = PBP^{-1}.$$

Theorem 42. *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues and algebraic multiplicities.*

Problems to be able to solve:

1. Find the characteristic polynomial of a matrix.
2. Solve problems similar to lecture.

5.3

Note: If $A = PDP^{-1}$ for P invertible, then $A^k = PD^kP^{-1}$.

Definition 43. *A square matrix A is said to be* ***diagonalizable*** *if A is similar to a diagonal matrix, i.e. if $A - PDP^{-1}$ where P is invertible and D is a diagonal matrix.*

Theorem 44 (Diagonalization)**.** *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover $A = PDP^{-1}$ were P is the n linearly independent eigenvectors that correspond to the diagonal entries of D which are the eigenvalues.*

Note: A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call this an **eigenvector basis** of $\mathbb{R}^n$.

Theorem 45. *An $n \times n$ matrix A is diagonalizable if it has n distinct eigenvalues.*

Theorem 46. *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$*

1. *For $1 \leq klep$, the dimensions of the eigenspace for λ_k is less than or equal to the algebraic multiplicity of λ_k.*

2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n. This*

3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_k$ forms an eigenvector basis for $\mathbb{R}^n$.*

Problems to be able to solve:

1. Compute A^k for diagonalizable A.

2. Diagonalize a matrix.

3. Determine if a matrix is diagonalizable.

4. Solve problems similar to lecture.

5.4

Definition 47. *Let V be a vector space. An* ***eigenvector*** *of a linear transformation $T : V \to V$ is a non-zero vector $\vec{x}$ in V such that $T(\vec{x}) = \lambda\vec{x}$ for some scalar λ. The scalar is called a* ***eigenvalue*** *of T if there is a nontrivial solution $\vec{x}$ of $T(\vec{x}) = \lambda\vec{x}$; such a vector is called an eigenvector corresponding to λ.*

Definition 48. *For a vector space V with basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, then the matrix for T relative to the basis $\mathcal{B}$ is*

$$M = [T9\vec{b}_1)]_{\mathcal{B}} \cdots [T(\vec{b}_n)]_{\mathcal{B}}].$$

Moreover for a vector $\vec{x}$ in V we have

$$[T(\vec{x}]_{\mathcal{B}} = M[\vec{x}]_{\mathcal{B}}.$$

Problems to be able to solve:

1. Find the matrix for a linear transformation T relative to a basis.

2. Solve problems similar to lecture.

Theorem 49 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. Then the following statements are equivalent:*

1. *A is an invertible matrix.*

2. *A is row equivalent to an $n \times n$ identity matrix.*

3. *A has n pivot positions.*

4. *$A\vec{x} = \vec{0}$ has only the trivial solution.*

5. *The columns of A form a linearly independent set.*

6. *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*

7. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.*

8. *The columns of A span $\mathbb{R}^n$*

9. *The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

10. *There is an* $n \times n$ *matrix* C *such that* $CA = I$.
11. *There is an* $n \times n$ *matrix* D *such that* $AD = I$.
12. A^T *is an invertible matrix.*
13. *Columns of* A *form a basis of* $\mathbb{R}^n$.
14. *Col* $A = \mathbb{R}^n$.
15. *rank* $A = n$.
16. *dim Nul* $A = 0$.
17. *Nul* $A = \{\vec{0}\}$.
18. $\det A \neq 0$.
19. *0 is not an eigenvalue of* A.
20. *The row space of* A *is* $\mathbb{R}^n$.