MATH 2641: Linear Algebra I

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Review:	Exam 1	**Place:**	6/17 - 6/18

1.1

Definition 1. *A **linear equation** in the variables* $x_1, \ldots, x_n$ *is an equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nc_n = b$$

where b *and the coefficients* $a_1, \ldots, a_n$ *are real or complex numbers.*

Definition 2. *A **system of linear equations** is a collection of one or more linear equations involving the same variables*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$$

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$$

$$\vdots$$

$$z_1x_1 + z_2x_2 + \cdots + z_nx_n = z$$

*A **solution** is a list* $(s_1, s_2, \ldots, s_n)$ *of numbers that makes each equation a true statement when substituted for their respective variable and a **solution set** is the set of all possible solutions. Lastly, we say two systems are **equivalent** if they have the same solution set.*

For solutions to a system, one of the following occurs:

1. There is no solution. System is **inconsistent**.
2. There is exactly one solution. System is **consistent**.
3. There is infinitely many solutions. System is **consistent**.

Definition 3.

1. *A **matrix** contains the essential information of linear system in a rectangular matrix where the coefficients of each variable align to columns of the matrix.*
2. *A matrix that contains only the coefficients on each variable is called the **coefficient matrix**,* $[A]$.
3. *The other matrix we consider is the **augmented matrix** which is the coefficient matrix together with an extra column that contains constants from the respective right sides of the equations,* $[A|b]$.
4. *The **dimension** of a matrix is* $m \times n$ *where* m *is the number of rows and* n *is the number of columns.*

Elementary Row Operations: Are used to turn a change a matrix into an equivalent matrix where the solutions are easier to find and they are the following three rules.

1. Replacement, replace one row by the sum of itself and a multiple of another row, $R_i + cR_j \to R_i$.

2. Interchange, interchange two rows, $R_i \leftrightarrow R_j$.
3. Scaling, multiply all entries in a row by a nonzero constant, $cR_i \to R_i$.

Problems to be able to solve:

1. Use elementary row operations to solve a system of linear equations.
2. Determine if a system is consistent or inconsistent.
3. Determine the condition for which a linear system is consistent.

1.2

Definition 4. *A rectangular matrix is in **echelon form** (REF) if it has the following three properties:*

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry (leftmost nonzero entry) is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zeros.*

*Further, if the matrix in echelon form satisfies the next two properties then it is in **reduced echelon form** (RREF):*

4. *The leading entry in each nonzero row is 1.*
5. *Each leading 1 is the only nonzero entry in its column.*

Echelon forms are not unique, the reduced echelon form is unique to each matrix.

Theorem 5. *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Definition 6. *Given a matrix in echelon form we have the following definitions:*

1. *A **pivot position** is the position of a leading entry in the echelon form of the matrix and is the same regardless the methods to get to the echelon form. These positions correspond to 1's in the reduced echelon form.*
2. *A **pivot** is a nonzero number that is used in a pivot position to create 0's or is changed to a leading 1, which in turn creates 0's.*
3. *A **pivot column** is a column of the matrix A that contains a pivot position.*

Definition 7. *A **basic variable** is a variable to corresponds to a pivot column in the augmented matrix of a system, and a **free variable** are all non-basic variables.*

Theorem 8. *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, meaning there is now row of the form*

$$[0 \quad \cdots \quad 0 \quad b]$$

Moreover, if the system is consistent and it has no free variables, then there is a unique solution. Otherwise, if there are free variables, then there are infinitely many solutions.

Problems to be able to solve:

1. Identify if a matrix is in REF or RREF.
2. Be able to row operations to find the RREF of a matrix.
3. Find the solution to an augmented matrix.
4. Determine if a system is consistent by looking at the matrix.

1.3

Definition 9. *A **vector** is a matrix with only one column. We say that two vectors are **equal** if and only if their corresponding entries are equal.*
Vector addition is component-wise, add the corresponding entries for each vector. Scalar multiplication by c, is multiplying each entry by c.

Note: You can think of vectors in $\mathbb{R}^n$ as points in $\mathbb{R}^n$ or directed line segments in $\mathbb{R}^n$.

Definition 10. *Given vectors $\vec{v_1}, \dots, \vec{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, \dots, c_p$, the vector $\vec{y}$ defined by*

$$\vec{y} = c_1\vec{v_1} + \cdots + c_p\vec{v_p}$$

*is called a **linear combination** of $\vec{v_1}, \dots, \vec{v_p}$ and **weights** $c_1, \dots, c_p$.*

Note: The vector equation

$$x_1\vec{a_1} + \cdots x_n\vec{a_n} = \vec{b}$$

has the same solution set as the augmented matrix whose columns are $\vec{a_1}, \dots, \vec{a_n}, \vec{b}$.

Definition 11. *Suppose the vectors $\vec{v_1}, \dots, \vec{v_p}$ are in $\mathbb{R}^n$, then the **span of the vectors***

$$span = \{\vec{v_1}, \dots, \vec{v_p}\}$$

is the set of all linear combinations of the vectors.

Note: In $\mathbb{R}^2$, the span of a single vector is the line through the origin. In $\mathbb{R}^3$ the span of two vectors is either a line through the origin or a plane.

Problems to be able to solve:

1. Compute sums and scalar products of vectors.
2. Convert between vector equations and a system of linear equations.
3. Determine if a vector is a linear combination
4. Characterize the span of set of vectors.

1.4

Definition 12. *If A is a $m \times n$ matrix, with columns $\vec{a_1}, \dots, \vec{a_n}$, and if $\vec{x}$ in $\mathbb{R}^n$, then the product of A and $\vec{x}$, denoted by $A\vec{x}$, is the linear combination of the columns of A using the entries of $\vec{x}$ as weights.*

$$A\vec{x} = [\ \vec{a_1} \quad \dots \quad \vec{a_n}\ ] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a_1} + \cdots + x_n\vec{a_n}$$

*called the **vector equation**.*

Theorem 13. *If A is an $m \times n$ matrix, with columns $\vec{a_1}, \dots, \vec{a_n}$ and if $\vec{b}$ is in $\mathbb{R}^m$, the matrix equation*

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\ \vec{a_1} \quad \cdots \quad \vec{a_n} \quad \vec{b}\ ]$$

Note: $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A, i.e. in the span of the columns.

Theorem 14. *Let A be an $m \times n$ matrix. Then the following statements are equivalent.*

1. *For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.*
2. *Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *A has a pivot position in every row.*

Theorem 15. *If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ and c a scalar, then:*

1. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2. $A(c\vec{u}) = cA\vec{u}$

Problems to be able to solve:

1. Compute products of a matrix and a vector.
2. Solve a matrix equation using augmented matrix.
3. Characterize the span of the columns of a matrix.
4. Answer questions involving Theorem 14.

1.5

Definition 16. *A system is called a* ***homogeneous system*** *if it can be written as $A\vec{x} = \vec{0}$. If it is not of this form, then it is called* ***nonhomogeneous****.*

Note: A homogeneous system always has the trivial solution $\vec{x} = \vec{0}$, however if there is at least one free variable in the RREF, then there are infinitely many solutions.

Theorem 17. *Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + v_h$ where v_h is the solution set to the homogeneous system.*

Problems to be able to solve:

1. Determine if a system has a nontrivial solution.
2. Solve a system and give the solution in its parametric form.
3. Use the number of pivots to determine the number of solutions.

1.6

Solve all the problem types we discussed.

1.7

Definition 18. *A set of* $\{\vec{v_1}, \ldots, \vec{v_p}\}$ *in* $\mathbb{R}^n$ *is said to be* ***linearly independent*** *if the vector equation*

$$x_1\vec{v_1} + \cdots + x_p\vec{v_p} = \vec{0}$$

has only the trivial solution. The set $\{\vec{v_1}, \ldots, \vec{v_p}\}$ *in* $\mathbb{R}^n$ *is said to be* ***linearly dependent*** *if there exists weights* $c_1, \ldots, c_p$ *not all zero such that*

$$c_1\vec{v_1} + \cdots + c_p\vec{v_p} = \vec{0}.$$

The equation above is called the ***linear dependence relation****.*

Note: The columns of a matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

Note: The linear dependence relation among the columns of A corresponds to a nontrivial solution to the matrix equation $A\vec{x} = \vec{0}$.

Note: If the RREF of the augmented matrix corresponding to $A\vec{x} = \vec{0}$ has a pivot in each column implies there are no free variables, so the columns of A are independent.

Note: If $A\vec{x} = \vec{0}$ has any free variables, then the columns of A are dependent.

Special Cases of Linear Independence

1. A set of one vector.
   (a) $\vec{v} \neq \vec{0}$, then it is linearly independent.
   (b) $\vec{v} = \vec{0}$, then it is linearly dependent.
2. A set of two vectors.
   (a) A set of two vectors is linearly independent if and only if neither vector is a multiple of each other.
   (b) A set of two vectors is linearly dependent if at least one vector is a multiple of the other.
3. A set containing the zero vector.
   (a) Theorem: A set of vectors $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ containing the zero vector is always linearly dependent.
4. A set with too many vectors.
   (a) Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 19. *An indexed set* $S = \{v_1, \ldots, v_p\}$ *of two or more vectors is linearly dependent if and only if at least one of the vectors in* S *is a linear combination of the others. In fact, if* S *is linearly dependent, and* $\vec{v_1} \neq \vec{0}$*, then some vector* $\vec{v_j}$*,* $j \geq 2$ *is a linear combination of the preceding vectors.*

Note: This last theorem says, if I can write a vector in my set as a combination of the others, then my set of vectors is linearly dependent.

Problems to be able to solve:

1. Determine if vectors are linearly independent through a matrix.
2. Determine if vectors are linearly independent through observation.
3. Determine if a vector is in the span of other vectors.
4. Find a nontrivial solution without performing row operations.

1.8

Definition 20. *A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector in $\vec{x}$ in $\mathbb{R}^n$ a vector $T(\vec{x})$ in $\mathbb{R}^m$. Here the **domain** of T is $\mathbb{R}^n$, $\mathbb{R}^m$ is the **codomain**, $T(\vec{x})$ is the image of $\vec{x}$ under T, and the set of all images is called the **range**.*

Note: A transformation that takes a matrix A and multiplies each vector $\vec{v}$ by A is called a matrix transformation, $\vec{x} \to A\vec{x}$.

Note: A matrix A that is $m \times n$ is a transformation from $T : \mathbb{R}^n \to \mathbb{R}^m$. The range of the transformation is all vectors that are linear combinations of the columns of A.

Definition 21. *If A is a $m \times n$, then the transformation $T(\vec{x}) = A\vec{x}$ has the following properties*

1. $T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A(\vec{u}) + A(\vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = A(c\vec{u}) = cA\vec{u} = cT(\vec{u})$

for all vectors in $\mathbb{R}^n$ and all scalars c.

Definition 22. *A transformation T is **linear** if it has the following properties*

1. $T(0) = 0$
2. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

for all vectors in $\mathbb{R}^n$ and all scalars c.

Note: The matrix $\begin{bmatrix} \text{r} & 0 \\ 0 & \text{r} \end{bmatrix}$ is a **contraction** if $0 \le r \le 1$, and a **dilation** when $r > 1$.

Problems to be able to solve:

1. Find the image of a vector under a linear transformation.
2. Use linearity to find the images of vectors under transformation.
3. Determine if a vector is in the range of a linear transformation.
4. Find matrices that define a given transformation.

1.9

Note: We can generalize the conditions for linearity

$$T(c_1\vec{v_1} + \cdots + c_p\vec{v_p}) = c_1T(\vec{v_1}) + \cdots + c_pT(\vec{v_p}).$$

Theorem 23. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$ for all vector in $\mathbb{R}^n$. In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\vec{e_j})$ where $\vec{e_j}$ is the jth column of the identity matrix in $\mathbb{R}^n$,*

$$A = [\ T(\vec(e_1)) \quad \cdots \quad T(\vec(e_n))\ ]$$

*and this is the **standard matrix for the linear transformation**.*

Definition 24. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$. Meaning the codomain of the transformation is equal to the range of the transformation.*

Definition 25. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$. Meaning that the equation $T(\vec{x}) = \vec{b}$ has either a unique solution or no solution.*

Theorem 26. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.*

Theorem 27. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation, and let A be the standard matrix for T. Then:*

1. *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A spans $\mathbb{R}^m$.*
2. *T is one-to-one if and only if the columns of A are linearly independent.*

Problems to be able to solve:

1. Find the standard matrix of a linear transformation.
2. Find vectors whose images under a linear transformation are given.
3. Determine if a transformation is one-to-one or onto.