The purpose of these notes is to provide a concise overview of the material covered in MATH 1001 at Georgia State University. Note that the chapters are presented in order that they are presented in the course due to exam scheduling.

# Chapter 11

This chapter covers some of the basic principles of combinatorics, including counting principles, permutations, and combinations. This section ends with some applications to probability.

## 11.1

The fundamental counting principle allows us to count how many options we have when picking one object from several different groups. In its simplest form, two choices, the fundamental counting principle is stated as follows:

**Theorem 1.** *If you can choose one item from a group of $M$ items and a second from a group of $N$ items, then the number of two item choices is $M \cdot N$.*

**Example 2.** *A restaurant offers six appetizers and twelve entrees. In how many ways can someone create a two course meal?*

*In this problem we have two choices to make, one for the appetizer which has 6 choices, and the second choice is for the entrees which has 12 choices. By the fundamental counting principle we have that*

$$\underline{6}_{\textit{appetizers}} \times \underline{12}_{\textit{entrees}} = 6 \times 12 = 72.$$

*Therefore there are 72 options for two course meals.*

Actually, we are able generalize the the fundamental counting principle to more than two choices in the following way:

**Theorem 3** (Fundamental Counting Principle)**.** *The number of ways in which a series of $M$ choices where choice one has $n_1$ choices, choice two has $n_2$ choices,..., and choice $M$ has $n_M$ choices, then the total number of ways we can make these $M$ choices is*

$$n_1 \times n_2 \times \cdots \times n_{M-1} \times n_M.$$

**Example 4.** *How many outfits can be made from two choices of shoes, three pairs of pants, and six shirts?*

*First, identify how many choices we have. We have three choices, shoes, pants, and shirts, so for the first choice shoes there are $n_1 = 2$ options, choice two pants has $n_2 = 3$ options, and the third choice shirts has $n_3 = 6$ options. Then by the fundamental counting principle there are*

$$\underline{2}_{\textit{shoes}} \times \underline{3}_{\textit{pants}} \times \underline{6}_{\textit{shirts}} = 2 \times 3 \times 6 = 36.$$

*Therefore there are 36 options of outfits.*

**Example 5.** *Dr. Titus is making a multiple choice exam containing eleven questions where each question has five choices with one correct answer per questions. If you select one answer per each question, leaving none of them blank, how many ways can you answer the questions?*

*First, we know that there are $M = 11$ choices and each choice has 5 options. Therefore by the fundamental counting principle there are*

$$\underline{5}_{\textit{Q1}} \times \underline{5}_{\textit{Q2}} \times \underline{5}_{\textit{Q3}} \times \underline{5}_{\textit{Q4}} \times \underline{5}_{\textit{Q5}} \times \underline{5}_{\textit{Q6}} \times \underline{5}_{\textit{Q7}} \times \underline{5}_{\textit{Q8}} \times \underline{5}_{\textit{Q9}} \times \underline{5}_{\textit{Q10}} \times \underline{5}_{\textit{Q11}} = 5^{11} = 48,828,125.$$

*Therefore there are 48,828,125 options on how to answer the exam.*

**Example 6.** *In the United States, telephone numbers are ten digits long, the first three digits are the area code and the remaining seven are called the local number. If an area code and local number cannot start with a 0 or 1, how many telephone numbers are there?*

*First we need to find out how many choices we have to make, and since a phone number is ten-digits long, then $M = 10$. Notice that for a phone number there are some restrictions in spots #1 and #4.*

$$\underline{\#1}_1 \underline{\#2}_2 \underline{\#3}_3 \qquad \underline{\#4}_4 \underline{\#5}_5 \underline{\#6}_6 \qquad \underline{\#7}_7 \underline{\#8}_8 \underline{\#9}_9 \underline{\#10}_{10}$$

*For spots not #1 and #4 there are 10 options for numbers that can appear in those spots, namely 0-9. In spots #1 and #4 we cannot have a 0 or 1, so there are only 8 options in these two places. Therefore by the fundamental counting principle there are*

$$\underline{8}_1 \times \underline{10}_2 \times \underline{10}_3 \times \underline{8}_4 \times \underline{10}_5 \times \underline{10}_6 \times \underline{10}_7 \times \underline{10}_8 \times \underline{10}_9 \times \underline{10}_{10} = 8^2 \times 10^8 = 6,400,000,000$$

*ways to create a phone number under these restrictions.*

## 11.2

Consider the following, we have six jokes to deliver in an order. If we have the stipulation that no joke can be repeated, how many different orders can we create? We can actually use the fundamental counting principle to answer it!

First note that we have six options which correspond to the order of each joke. Consider the following spots:

1. In spot one there are no restrictions, we have 6 jokes to pick from.
2. In spot two, there is one restriction, we cannot pick to joke in spot # 1. Thus there are 5 options in spot #2.
3. In spot three, there are two restrictions, namely that we cannot pick the jokes in spots #1 and #2. Thus there are 4 options in spot #3.
4. In spot four, there are three restrictions, namely that we cannot pick the jokes in spots #1 , #2 and #3. Thus there are 3 options in spot #4.
5. In spot five, there are four restrictions, namely that we cannot pick the jokes in spots #1, #2, #3, and #4. Thus there are 2 options in spot #5.
6. In spot six, there are five restrictions, namely that we cannot pick the jokes in spots #1, #2, #3, #4, and #5. Thus there is 1 option in spot #6.

Therefore, by the fundamental counting there are

$$\underline{6}_{\#1} \times \underline{5}_{\#2} \times \underline{4}_{\#3} \times \underline{3}_{\#4} \times \underline{2}_{\#5} \times \underline{1}_{\#6} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

We can tell the six jokes in 720 distinct ways. Note that since ordering of the jokes matters, these are all different!

These arrangement of jokes have a special name, called **permutations**.

**Definition 7.** *A **permutation** is an ordered arrangement where no item is used more than once and most importantly the order of the items makes a difference.*

**Example 8.** *Consider the same six jokes with the names A, B, C, D, E, and F. How many distinct orders can we make if joke F must be delivered first and joke C must be delivered third.*

*Similar to the earlier example, there are 6 choices that have to be made, however we have restrictions.*

1. *In spot one there is one restrictions, we have to have joke F in the first spot. Thus we have 1 option in spot one.*
2. *In spot two, there is one restriction, we cannot pick to joke in spot # 1. Thus there are 5 options in spot #2.*
3. *In spot three, there are two restrictions, namely that we cannot pick the jokes in spots #1 and #2. Thus there are 4 options in spot #3.*

4. *In spot four there is one restrictions, we have to have joke C in the fourth spot. Thus we have 1 option in spot four.*
5. *In spot five, there are four restrictions, namely that we cannot pick the jokes in spots #1, #2, #3, and #4. Thus there are 2 options in spot #5.*
6. *In spot six, there are five restrictions, namely that we cannot pick the jokes in spots #1, #2, #3, #4, and #5. Thus there is 1 option in spot #6.*

*Therefore, by the fundamental counting there are*

$$\underline{1}_{\#1} \times \underline{4}_{\#2} \times \underline{3}_{\#3} \times \underline{1}_{\#4} \times \underline{2}_{\#5} \times \underline{1}_{\#6} = 1 \times 4 \times 3 \times 1 \times 2 \times 1 = 24.$$

*We can tell the six jokes in 24 distinct ways.*

**Example 9.** *You want to arrange 7 books on a shelf. If order of the books matters, how many ways can you arrange the seven books?*

*If we arrange the books from left to right, then we make seven choices for each spot. Therefore, by the fundamental counting there are*

$$\underline{7}_{\#1} \times \underline{6}_{\#2} \times \underline{5}_{\#3} \times \underline{4}_{\#4} \times \underline{3}_{\#5} \times \underline{2}_{\#6} \times \underline{1}_{\#7} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$$

*We can arrange the seven books in 5040 distinct ways.*

The product in the last example, $7 \times 5 \times 4 \times 3 \times 2 \times 1$, has a special name and symbol. In this case it is called 7 factorial and is represented by 7!.

**Definition 10.** *For any positive integer $n$, $n!$, is the product of $n$ down through 1,*

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

*We also define $0!$ to equal 1.*

**Example 11.** *Evaluate $\frac{8!}{5!}$ and $\frac{26!}{21!}$*

$$\frac{8!}{5!} = \frac{(8)(7)(6)(5)(4)(3)(2)(1)}{((5)(4)(3)(2)(1)} = (8)(7)(6) = 336.$$

$$\frac{26!}{21!} = \frac{(26)(25)(24)(23)(22)(21!)}{21!} = (26)(25)(24)(23)(22) = 7,893,600.$$

The nice thing is that there is a formula for calculating the number of permutations using the factorial. More precisely,

**Theorem 12.** *The number of possible permutation if $r$ items are taken from $n$ items is*

$$_nP_r = \frac{n!}{(n-r)!}.$$

**Example 13.** *Your group is choosing three officers out of twenty members. The group is picking a CEO first, manager second, and secretary third. If the positions are chosen in that order and no one can hold more than one office, how many ways can officers be chosen?*

*Note that for this scenario that order does matter and we are not allowed to repeat picking a person, so this is a problem asking us to calculate permutations. Thus we are picking 3 people from a group of 20, i.e. $r = 3$ and $n = 20$.*

$$_{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 20 \times 19 \times 18 = 6840.$$

*Meaning that there are 6840 distinct ways to chose the three officers.*

**Example 14.** *We have a seven member board and we want to count the number of ways we can choose a President, Vice-President, Secretary, and Treasurer assuming no one can have more than one position.*

*This is very similar to the previous example since we can think of our choices as ordered, meaning the president is chosen first, vice-president second, and so on. Further, since no one can have more than one position, we are dealing with permutations. Thus we are picking 4 people from a group of 7, i.e. $r = 4$ and $n = 7$.*

$$_4P_7 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 7 \times 6 \times 5 \times 4 = 840.$$

*Meaning there are 840 distinct ways to pick the four officers.*

Now, consider the letters ANA, how many ways can we rearrange the letters uniquely? Now, if we are able to differentiate the two A's from each other there are $3! = 6$ ways.

ANA, AAN, NAA, NAA, ANA, AAN

However, if we are not able to differentiate the A's, then there are only 3 options, ANA, AAN, and NAA.

Luckily we can deal with these permutations with duplicate items by

**Theorem 15.** *The number of permutations of $n$ items where $p$ are identical, $q$ are identical, $r$ are identical, and so on is*

$$\frac{n!}{p!q!r!\cdots}$$

Note that in the last example we had $n = 3$ letters and $p = 2$ are identical for the A's, thus there are

$$\frac{3!}{2!} = 3$$

distinct arrangements of ANA.

**Example 16.** *How many distinct ways can we rearrange the letters in MISSISSIPPI distinctly?*

*This is a permutation of $n$ items with repetitions, so we need to find our values. There are $n = 11$ items, $p = 4$ repetitions of $i$, $q = 4$ repetitions s, and $r = 2$ repetition of p. Therefore the number of distinct rearrangements is*

$$\frac{n!}{p!q!r!} = \frac{11!}{4!4!2!} = 34,650.$$

## 11.3

**Definition 17.** *A **combination** occurs when items are selected from the same groups, no item is used more than once, and the order of the items does not matter.*

The big thing to understand is the differences between combinations and permutations. Remember that for a permutation the order matters and for combination has that order does not matter.

**Example 18.** *Identify if the following scenario is a permutation or combination.*

1. *We have six students running for the positions president, vice-president, and treasurer. The highest number of votes is the president, the second highest is vice-president, and the third highest is treasurer. How many different boards can we create?*

   *This is a permutation, considering this is equivalent to picking with the order president, vice-president, and treasurer. Thus order does matter.*

2. *There is a board of seven supervisors. A three person committee is needed to do a study, how many different committees can be formed?*

   *This is a combination problem since there are no titles within the committee, so order does not matter.*

3. *Baskin Robbins offers thirty-one flavors. Since one of the menu items is a three scoop bowl, each a different flavor, how many different bowls can be created?*

   *This is a combination problem since the order that the ice cream is placed in a bowl does not matter.*

The notation we will use for combinations is ${}_nC_r$, meaning that the number of combinations of $n$ things taken $r$ at a time, i.e. $n$ choose $r$.

Interesting enough, if we have a collection of $r$ items from $n$, there are $r!$ permutations for this collection, so the formula for combinations is given by

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!}.$$

**Example 19.** *How many three person committees can be created from eight people?*

*Note this is a combination problem since order does not matter. Moreover, we are choosing three people from eight, i.e. $r = 3$ and $n = 8$. Thus there are*

$${}_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = 56$$

*ways to form the committee.*

**Example 20.** *How many choices of pet combinations if picking four animals out of seven.*

*Note this is a combination problem since order does not matter. Moreover, we are choosing fours pets from seven, i.e.* $r = 4$ *and* $n = 7$. *Thus there are*

$$_7C_4 = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = 35$$

*ways to form the animal council.*

**Example 21.** *The United State senate is comprised of 46 Democrats, 52 Republicans, and 2 Independent senators. How many distinct five person committees can be created if each committee must have two Democrats and three Republicans?*

*For this example we want to use the fundamental counting principle since I have two choices to make, one for the Democrats and one for the Republicans.*

*First, how many ways can I pick two Democrats out of 46? Well, 46 choose 2, i.e.*

$$_{46}C_2 = \frac{46!}{(46-2)!2!} = \frac{46!}{44!2!} = 1035.$$

*Secondly, how many ways can I pick three Republicans out of 52? Well, 52 choose 3, i.e.*

$$_{52}C_3 = \frac{52!}{(52-3)!3!} = \frac{52!}{49!3!} = 22,100.$$

*It then follows from the fundamental counting principle that the number of five person committees with these restrictions is*

$$(1035) \times (22,100) = 22,873,500.$$

*Note this is just the product* $_{46}C_2 \times_{52} C_3$*!*

**Example 22.** *If there are six male bears and seven female bears at the zoo. How many combinations can be made it two males and three females must be selected?*

*This again is an application of the fundamental counting principle and combinations. Thus*

$$_6C_2 \times_7 C_3 = 525.$$

## 11.4

You toss a coin in the air, it is equally likely to land either heads up, H, or tails up T, even though the actual outcome is uncertain.

Any occurrence where the outcome is uncertain is called an **experiment**. The set of all possible outcomes from an experiment is called the **sample space**, denoted $S$. In the case of a coin flip

$$S = \{H, T\}.$$

We say that an **event** is any subset of the sample space, usually denoted $E$. For example, the event of flipping a head is $E = \{H\}$.

If an event $E$ has $n(E)$ equally likely outcomes and the sample space has $n(S)$ likely outcomes, the probability of event $E$, denoted $P(E)$ is

$$P(E) = \frac{\text{\# of outcomes in an event E}}{\text{total possible outcomes}} = \frac{n(E)}{n(S)}.$$

Thus the probability of flipping a head is given by $n(E) = 1$ (only one item in the event $E$) and $n(S) = 2$ since the outcomes are either H or T,

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}.$$

**Example 23.** *A six-sided die is rolled once. Find the following probabilities.*

*Note that* $S = \{1, 2, 3, 4, 5, 6\}$ *so* $n(S) = 6$.

1. *Rolling a 3. Here* $E = \{3\}$ *and* $n(E) = 1$*, so the probability of rolling a 3 is*

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

   *Note this is the same for any number 1-6.*

2. *Rolling an even number. Here* $E = \{2, 4, 6\}$ *and* $n(E) = 3$*, so the probability of rolling an even number is*

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

3. *Rolling a number less than five. Here* $E = \{1, 2, 3, 4\}$ *and* $n(E) = 4$*, so the probability of rolling a number less than 5 is*

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}.$$

4. *Rolling a number less than ten. Here* $E = \{1, 2, 3, 4, 5, 6\}$ *and* $n(E) = 6$*, so the probability of rolling a number less than ten is*

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{6} = 1.$$

5. *Rolling a number larger than six. Here* $E = \{\}$ *and* $n(E) = 0$*, so the probability of rolling a number larger than six is*

$$P(E) = \frac{n(E)}{n(S)} = \frac{0}{6} = 0.$$

**Example 24.** *You are dealt one card from a standard 52-card deck. Find the following probabilities.*

*Note* $n(S) = 52$*.*

1. *Drawing a king. There are 4 kings, one for each suit, so* $n(E) = 4$*. Thus* $P(E) = 4/52$*.*
2. *Drawing a heart. There are 13 cards with the suit of heart, so* $n(E) = 13$*. Thus* $P(E) = 13/52$*.*
3. *Drawing a king of hearts. There is only 1 card fitting these restrictions, so* $n(E) = 1$*. Thus* $P(E) = 1/52$*.*

Sometimes we will deal with what is called empirical probability, which is probability determined by counting the number of times an event is observed. Thus the empirical probability of an event $E$ is

$$P(E) = \frac{\text{observed number of times the event occurs}}{\text{total number of observed occurrences}}.$$

Note that the bottom is simply the number of times we ran the experiment.

**Example 25.** *We surveyed approximately 254 million Americans ages fifteen and older and asked them about their marital status. The following results are expressed in millions.*

| | *Married* | *Never Married* | *Divorced* | *Widowed* | *Total* |
|---|---|---|---|---|---|
| *M* | *66* | *43* | *11* | *3* | *123* |
| *F* | *67* | *38* | *15* | *11* | *131* |
| *Total* | *131* | *81* | *26* | *14* | *254* |

*What is the probability of selecting a divorced person if you select a random American.*

*Since there are 26 million Americans that are divorced, then* $n(E) = 26$ *and out of* $n(S) = 254$*. Thus*

$$P(E) = \frac{26}{254} \approx .10.$$

*What is the probability of selecting a never married male?*

*Since there are 43 never married men, we have that* $n(E) = 43$*. Thus*

$$P(E) = \frac{43}{254}.$$